Part I

Erwin Bolthausen: Large Deviations and Interacting Random Walks

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Introduction

The lectures presented here treat three closely related topics on random walks with self-interactions or with interactions with a wall. In some cases, the "random walk" is a Brownian motion. All the topics have versions for random walks *and* for Brownian motion, but not in all cases both versions have been proved.

The first topic addressed is the so called polymer measure in three dimension, also called Edwards' model. This is a construction of a Brownian motion with local self-repellence, which is given by a δ-function. There are considerable difficulties to define this rigorous for dimensions larger than 1. The interaction in terms of this δ-function is not defined at all. One then tries to work with a regularized version, for instance smoothing the δ-function, and removes the regularization in a limiting procedure, proving that the limit measure exist. The first construction in the difficult three-dimensional case was by Westwater who in two celebrated papers in the early eighties proved that a suitably regularized version converges if the regularization is taken away. The two-dimensional case is easier and had been treated earlier by Varadhan. Westwater's approach is extremely complicated, and essentially nobody seems to haven taken the pains to study his papers and his methods. Not much later, there has been an alternative approach, first in the context of quantum field theory, by Brydges, Fröhlich and Sokal, and a bit later also for the polymer problem by Bovier, Felder and Fröhlich. Their approach is *much* simpler, but also had considerable shortcomings. The main one was that it was impossible to speak of *the* polymer measure, as the proof gave only boundedness properties of finite dimensional distributions of (lattice) regularized versions, from which the existence of convergent subsequences could be derived. With such a method, it is then difficult to prove important properties of the measure. In a paper of mine in 1992, most of these shortcomings have been removed, and the convergence of a regularized version has been proved. The topic was vigorously taken up by X.Y. Zhou who wrote a number of papers (mostly with Albeverio, and one with me), extending the approach for instance to arbitrary coupling constants, identifying the measure with the one constructed by Westwater, and proving limit theorems for self-repellent discrete random walks converging to the measure. It was an extremely sad event when Zhou died suddenly in 1996.

The second chapter will deal with self-attracting random walks. The self-attraction is given, too, in terms of a Gibbsian description, which contrasts with some models investigated recently in the probability community. The most natural example would be to change the sign of the coupling constant in the standard (weakly) self-avoiding case, but it is easy to see that this is not an interesting object as the attraction would be far too strong. So one is led to models with weaker interaction, namely where the coupling constant decays in time. Somewhat surprisingly, this model has a collapse transition in two and more dimensions, changing from a collapsed state when the coupling constant is large, to a diffusive one for weak couplings. The diffusive phase has been studied by Brydges and Slade. The collapsed state is investigated in a paper of Uwe Schmock and myself. Collapse transitions are well known in statistical mechanics in many models with self-attraction, but in most cases, there is no rigorous proof.

A part of the second chapter will be devoted to problems around the Wiener sausage, where closely related effects appear. A self-attracting path measure (starting for instance with the Brownian motion) is obtained by transforming the measure by favoring paths with small Wiener sausage. It turns out that the path measure obtained in this way, leads to a kind of droplet construction, where the droplet describes in which region of the space the paths have to concentrate under the new measure. This droplet is somewhat trivial, being just a ball, a fact which is related to the standard isoperimetric problem. Recently, after previous work in the two-dimensional case by Sznitman and by myself, Povel has been able to prove in any dimension that the droplet concentrates in L_∞ near the optimal droplet. The behavior of this model also depends crucially on the coupling constant chosen. It turns out that a model with decaying coupling constant is just diffusive if the decay is too fast. There is a critical case where the “droplet picture” starts to dissolve, which is quite interesting, and has recently been investigated by M. van den Berg, F. den Hollander and me. I will present the main ideas on this topic.

The second chapter circles around models which have a localization-delocalization transition, and this topic is continued in the last chapter which discusses two models with localization-delocalization phenomena of a somewhat different kind, namely coming from an interaction of a random walk with a “wall”. The chapter covers a model of a so called hetero- or copolymer with a localization-delocalization phase transition, and furthermore a so-called wetting transition in dimension one.

Some comments about the degree in which technical details will be given in these lectures. Some of the proofs presented here would be technically very lengthy if given in all details. For instance, a full and complete proof of the construction of the three dimensional polymer measure would still require considerable space, but in fact, some of the calculations and estimates are quite repetitive, and it would only be tiring if all of them would be presented. As a rule, I am trying to present for most of the results some of the very core arguments in details. I will furthermore essentially concentrate on the probabilistic aspects, just citing the analytic ones.

Many parts of these lectures can be read independently. In particular, the first chapter on the three-dimensional polymer measure stands somewhat apart. The other parts are all closely connected with large deviations (Chapter 1 actually, too, but somewhat hidden).

1. On the construction of the three-dimensional polymer measure

1.1 Introduction

An outstanding open problem in probability theory is the determination of the mean end to end distance of a standard self-avoiding random walk on the d-dimensional lattice $\mathbb{Z}^d$ for $d = 2, 3$ (and 4).

Given $n \in \mathbb{N}$, let Ω_n be the set of paths ω of length n:

$$\Omega_n \stackrel{\text{def}}{=} \{\omega = (\omega_0, \omega_1, \ldots, \omega_n) : \omega_i \in \mathbb{Z}^d,\, \omega_0 \\ = 0,\, |\omega_i - \omega_{i-1}| = 1 \,\text{for}\, 1 \leq i \leq n\},$$

and the set of self-avoiding paths

$$\Omega_n^{SA} \stackrel{\text{def}}{=} \{\omega \in \Omega_n : \omega_i \neq \omega_j \text{ for } i \neq j\}.$$

The main problem is to derive precise information about the asymptotic behavior of $|\Omega_n^{SA}|$, the number of self-avoiding paths, and about the mean length of self-avoiding paths:

$$\langle \|\omega_n\| \rangle_{SA} \stackrel{\text{def}}{=} \sum_{\omega \in \Omega_n^{SA}} \|\omega_n\| / |\Omega_n^{SA}|$$

where $\| \quad \|$ is the Euclidean length. From arguments in theoretical physics (conformal field theory, expansion techniques) it is believed that $\langle \|\omega_n\| \rangle_{SA}$ scales with $n^{3/4}$ for $d = 2$, and with n^ν with ν slightly less than 3/5 for $d = 3$. Also, the scaling limits, i.e. the asymptotic distribution of $\omega_n / \langle \|\omega_n\| \rangle_{SA}$ should be non Gaussian (see [52]). From dimension 4 onwards, the scaling limits are becoming Gaussian, with a slight correction to ordinary central limit scaling for $d = 4$, where $\langle \|\omega_n\| \rangle_{SA}$ is believed to be of order $\sqrt{n}\sqrt[8]{\log n}$. The case of $d \geq 5$ is completely settled: Starting with work by Brydges and Spencer [25] who introduced the lace expansion, and culminating with Hara and Slade [55]. An excellent monograph on these and related topics is [59]. (For a recent conceptually simple approach, see [18]). There is no (published) proof for $d = 4$ which is not (directly) tractable by lace expansions (see [22], [58], [26] for partial results).

I will not give any discussion of these techniques here. One of the results I discuss is a very weakly interactive case for $d = 3$, where the interaction is so weak that one has ordinary scaling, but where nevertheless the scaling limit of $(\omega_{[nt]}/\langle\|\omega_n\|\rangle)_{0\le t\le 1}$, which is shown to exist, is not Gaussian, but instead given as the so called Edwards' model first constructed rigorously by Westwater [76].

We now introduce the so called weakly self-avoiding random walks. Here, all paths in Ω_n receive positive weight, but the ones with many intersections are "punished". This is achieved by choosing a parameter $\lambda \in (0,1)$. Then every path $\omega \in \Omega_n$ gets its relative weight decreased by a factor $(1-\lambda)$ for every self intersection, i.e. we define the probability measure on Ω_n by

$$\hat{P}_{n,\lambda}(\omega) \stackrel{\text{def}}{=} \mathop{\Pi}_{0\le i<j\le n} (1-\lambda 1_{\omega_i=\omega_j}) \;/\; Z_{n,\lambda},$$

where $Z_{n,\lambda} \stackrel{\text{def}}{=} \sum_{\omega\in\Omega_n} \mathop{\Pi}_{0\le i<j\le n} (1-\lambda 1_{\omega_i=\omega_j})$.

(Remark: Through these notes, we will always use $\hat{P}$ to denote measures on path spaces obtained from "simple" random walk measures by introducing interactions, self-repelling in this chapter, and self-attracting in the next.) We rewrite the above measure by setting (with a slight abuse of notation)

$$\hat{P}_{n,\beta}(\omega) = \exp\left[-\frac{\beta}{2}\sum_{i,j=1}^{n} 1_{\omega_i=\omega_j}\right] \Big/ Z_{n,\beta}, \tag{1.1}$$

where $\beta = -\log(1-\lambda) \in (0,\infty)$, and $Z_{n,\beta}$ being the appropriate norming. Remark that the diagonal part in the summation is cancelling. We can also rewrite the interaction:

$$\sum_{i,j=1}^{n} 1_{\omega_i=\omega_j} = \sum_{x\in\mathbb{Z}^d} \ell_n(x,\omega)^2,$$

where $\ell_n(x,\omega)$ is the discrete local time

$$\ell_n(x,\omega) \stackrel{\text{def}}{=} \sum_{j=0}^{n} 1_{\omega_j=x}.$$

This is the so called Domb-Joyce model.

The above expression for the Domb-Joyce model naturally leads to the question if similar models exist starting with the Brownian motion instead of the random walk and how the relations between this and the discrete models are.

We start with the Wiener measure P_T on $C_0^d(T)$, the set of continuous paths $\omega : [0,T] \to \mathbb{R}^d$, starting at 0, and we want to define the polymer measure formally by

$$\hat{P}_{T,\beta}(d\omega) = \frac{1}{Z_{T,\beta}} \exp\left[-\frac{\beta}{2}\int_0^T dt \int_0^T ds\ \delta(\omega_t-\omega_s)\right] P_T(d\omega), \tag{1.2}$$

where δ is the Dirac function. There is evidently some trouble defining this, as the formal expression

$$\int_0^T ds \int_0^T dt\ \delta(\omega_t - w_s) = \int dx\ \ell_T(x,\omega)^2,$$

where

$$\ell_T(x,\omega) = \int_0^T \delta(\omega_s - x) ds,$$

i.e. the L_2-norm of the local time, only makes sense for $d = 1$. The trouble is also revealed by formally calculating the expectation under Wiener measure

$$E_T \int_0^T ds \int_0^T dt\ \delta(\omega_t - \omega_s) = 2 \iint_{0 \le s \le t \le T} dsdt\ p_{t-s}(0), \tag{1.3}$$

where $p_u(x)$ is the transition density of Brownian motion, i.e.

$$p_u(x) \stackrel{\text{def}}{=} (2\pi u)^{-d/2} e^{-\frac{\|x\|^2}{2u}}.$$

However, the right hand side of (1.3) is evidently divergent for $d \ge 2$. There are a number of ways in which one can try to remedy the situation. The first idea, but not the easiest one, is to step back to the Domb-Joyce model and to try to make some limiting procedures with the lattice spacing going to 0, and an appropriate dependence of β on n. This is possible, but is somewhat delicate, and has only recently been done in a completely satisfactory way [1] for $d = 3$. I will discuss that below. Another approach is to replace δ by a smoothed version, e.g. $p_\varepsilon, \varepsilon > 0$, and then let $\varepsilon \to 0$. For $d = 2$ this was the way in which Varadhan proved the existence of the polymer measure (1.2). The most convenient way however is to use some gap regularization. Observe that the right hand side of (1.3) is divergent only because of the integration near the diagonal. If we leave a gap between s, t, e.g. integrating only over $s + \varepsilon \le t,\ \varepsilon > 0$, then this stays finite. It is in fact known that

$$J_{0,T}^{\varepsilon}(\omega) = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^T dt\ \delta(\omega_t - \omega_s)$$

is well defined, P_T - a.s. As this is still only a formal expression, some comments are in order. We can define, for every $a > 0$,

$$J_{0,T}^{\varepsilon,a}(\omega) = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^T dt\ p_a(\omega_t - \omega_s),$$

and then (with fixed $\varepsilon > 0$) let $a \to 0$. This limit exists e.g. in L_2 (see [64]), and is what we denote by $J_{0,T}^{\varepsilon}$. The limit has nice properties, e.g. it is a.s.

continuous in ε, T. We will not go into a discussion of these properties, but simply refer to the relevant literature, e.g. [64]. We then define our regularized Edwards' model by

$$\hat{P}^{\varepsilon}_{T,\beta}(d\omega) = \exp(-\beta J^{\varepsilon}_{0,T}(\omega)) P_T(d\omega)/Z_{T,\beta,\varepsilon}. \tag{1.4}$$

Theorem 1.1. *For $T, \beta > 0$, $d = 2, 3$, the limit*

$$\hat{P}_{T,\beta} = \lim_{\varepsilon \to 0} \hat{P}^{\varepsilon}_{T,\beta}$$

exists as a weak limit of probability measures on $C^d_o(T)$.

We will focus on the case $d = 3$ which is considerably more delicate than the case $d = 2$. The theorem is essentially due to Westwater [76]. The only difference is that he took a slightly different gap regularization. The procedure Westwater follows is, however, extremely difficult. We give some comments on it below. We explain here an approach which is *much* easier and is based on so called skeleton inequalities. This method had been introduced by Brydges, Fröhlich and Sokal [23] in Euclidean φ^4_d quantum field theory and had then been adapted to the polymer problem in [19]. From a probabilistic point of view, the results in [19] have however a number of shortcomings. The most serious one is that no convergence is proved, but only boundedness properties which made it possible to prove the existence of convergence subsequences. There was then no possibility for an identification of the process for instance with the one constructed by Westwater. The above authors also had used a lattice regularization, and it was not even clear that there are limits which are rotational symmetric. Another point is that the results are formulated in terms of Laplace transforms in time, and the problem how to get polymer measures with fixed time horizon had not been addressed. For this reason, the approach was generally thought to be simple but that it would give only somewhat weak results. However, at least for the polymer case, some of the shortcomings can be remedied, and a modification proving convergence at fixed time has been developed in [9]. This is explained below.

In [9], the polymer measure was constructed only for a small coupling parameter β. This restriction has later been removed in [2], by a simple but clever argument, which we will include here, too.

First some comments on the approach by Westwater [76]. He uses no regularization of the δ-function but a slightly different gap regularization than that explained above. Take $T = 1$ (for notational simplicity). Then

$$X_0(\omega) = \int_0^{1/2} ds \int_{1/2}^1 dt \ \delta(\omega_t - \omega_s)$$

is P_1-a.s. well defined (in the sense described above). On the next level, one defines

$$X_{1,1}(\omega) = \int_0^{1/4} ds \int_{1/4}^{1/2} dt\ \delta(\omega_t - \omega_s),$$

$$X_{1,2}(\omega) = \int_{1/2}^{3/4} ds \int_{3/4}^{1} dt\ \delta(\omega_s - \omega_t),$$

and then of course

$$X_{n,i}(\omega) = \int_{(i-1)2^{-n}}^{(2i-1)2^{-n-1}} ds \int_{(2i-1)2^{-n-1}}^{i2^{-n}} dt\ \delta(\omega_s - \omega_t),$$

$n \geq 1, 1 \leq i \leq 2^n$. These variables have a number of simple properties. For fixed n, the 2^n variables $X_{n,i}$ are evidently independent. Furthermore, the law of $X_{n,i}$ by simple Brownian rescaling is the same as that of $X_{n-1,i}/\sqrt{2},\ n \geq 2$ for $d = 3$. The *main* difficulty is that for different n, the $X_{n,i}$ are not independent. Westwater proves that there is a near independence between $X_{n,i}$ and $X_{m,i}$ if $|m-n|$ is large. In other words, there is near independence between short and long range self intersections. Westwater then proves, using this property, that $\lim_{N\to\infty} \hat{P}_{1,\beta}^{N,WW}$ exists where

$$\hat{P}_{1,\beta}^{N,WW}(d\omega) = \exp\left[-\beta \sum_{n=0}^{N} \sum_{j=1}^{2^n} X_{n,j}(\omega)\right] P_1(d\omega)/Z_\beta^N.$$

The picture below (Fig. 1.1) shows the domain of integration for $N = 2$. The main disadvantage of the Westwater approach is that it is extremely complicated which is mainly due to the fact that it makes bad use of the fact that $X_{n,i} \geq 0$. A further enormous complication arises because the $X_{n,i}$ do not have exponential moments. It has recently been proved by Albeverio and Zhou [2] that the Westwater process coincides with the one of Theorem 1.1. This might look obvious, but in fact, the removal of the gap is quite subtle as will become apparent.

One of the motives to investigate the continuous polymer measures had certainly been the hope that they shed some light on the discrete model. The relation is however quite delicate. To see what the appropriate scaling should be, we will perform some formal calculations. We consider the polymer measure on a time slot $[0, T]$ with a coupling parameter $\beta_T > 0$ which may depend on T. Formally

$$d\hat{P}_{T,\beta_T} = \exp\left[-\beta_T \int_0^T ds \int_s^T dt\ \delta(\omega_t - \omega_s)\right] P_T(d\omega)/Z.$$

Performing Brownian scaling

$$\tilde{\omega}_t = \omega_{tT}/\sqrt{T},\ t \leq 1,$$

and using

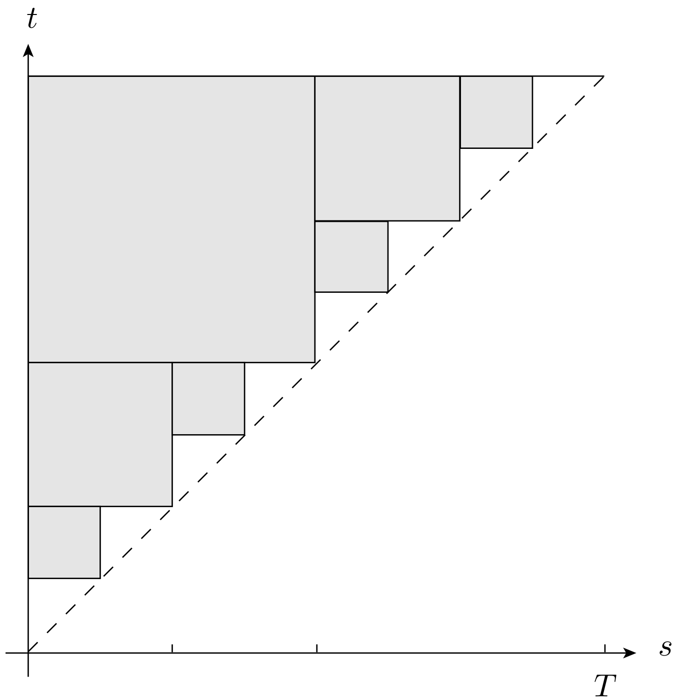


Fig. 1.1.

$$\int_0^T ds \int_s^T dt \; \delta(\omega_t - \omega_s) = T^{2-\frac{d}{2}} \int_0^1 ds \int_s^1 dt \; \delta(\tilde{\omega}_s - \tilde{\omega}_t)$$

we see that for $\beta_T = \beta T^{\frac{d}{2}-2}$, the distribution of the rescaled path under $\hat{P}_{T,\beta_T}$ is just $\hat{P}_{1,\beta}$. (This is of course not a rigorous proof, but the statement is correct). Anyway, this suggests that starting with a standard random walk $(\omega_0 = 0, \omega_1, \ldots, \omega_T)$ on $\mathbb{Z}^d$, and the weakly self-avoiding walk

$$\tilde{P}_{T,\beta}(\omega) = \frac{1}{Z_{T,\beta}} \exp\left[-\beta \sum_{0 \le i < j \le T} 1_{\omega_i = \omega_j}\right].$$

one has

Theorem 1.2. *Assume $d \le 3$ and $\beta > 0$. Then*

$$\lim_{T \to \infty} \tilde{P}_{T,\beta T^{-2+d/2}} Y_T^{-1} = \hat{P}_{1,\beta}$$

where $Y_T : \Omega_T \to C_o^d(1)$ is defined by $Y_T(\omega)(i/T) = \omega_i/\sqrt{T}$, and linearly interpolated between.

The above Theorem is easy for $d = 1$, has been proved by Stoll [69] for $d = 2$, and in [1] for $d = 3$. It is of course far from the "real" question, namely what happens with $\tilde{P}_{T,\beta}$ for fixed β as $T \to \infty$. On the other hand, even in the above "very weakly" self-avoiding case, the limiting measure for $d = 3$ is singular with respect to any Wiener measure, and is non-Gaussian, as has been proved by Westwater. Remark that in the two dimensional case, the T-dependence of β_T is $\beta_T = \beta/T$. This will be important in Chapter II.

There are considerable technical difficulties to prove Theorem 1.2 for $d = 3$. The main problem is to show that the short range intersections, where the random walk does not quite look like a Brownian motion, do not disturb the limiting picture. We will not give a detailed proof here. It is essentially a modification of the arguments in the proof of Theorem 1.1 but requires some additional nontrivial arguments.

It is to be expected that the limiting behavior of the weakly self-avoiding model (i.e. $\tilde{P}_{T,\beta}$ for fixed β, $T \to \infty$) is by the above rescaling related to the $\beta \to \infty$ behavior of the polymer measure $\hat{P}_{1,\beta}$. There is no proof of this for $d \geq 2$. Even the $d = 1$ case is very far from trivial, and has only recently been solved by van der Hofstad and den Hollander [57].

We give an outline of the rest of this chapter. We entirely focus on $d = 3$ which is the most delicate case. In Section 1.2 we discuss the boundedness properties of the so called two point functions. This follows closely the approach in [23] and [19], but there are some differences. First, we avoid using Laplace transforms in time. Proving things in Laplace transformed versions is technically simpler but then one has the trouble to invert the result. This inversion is not done in the above mentioned papers of Brydges, Fröhlich, Sokal and Bovier, Felder and Fröhlich. We also derive relatively sharp pointwise estimates (in contrast to L_p-estimates).

I believe that in the long run, methods for fixed time and directly in x-space give sharper results and are more transparent than when using various transforms which have to be inverted. Another example of this is a recent direct approach to weakly self-avoiding walks for $d \geq 5$ [18], which is conceptually much simpler than older ones and yields somewhat sharper results.

I will present some details of the proof of Theorem 1.1, but not all. First, I take some continuity properties of intersection local times for granted. These are modifications of classical results proved by Rosen. For details I refer to the Appendix of [9]. The basic inequalities are explained in details but the calculations are somewhat repetitive and I will not give all of them.

The boundedness properties immediately imply the tightness of the measures, as $\varepsilon \to 0$. With the inequalities derived in 1.2, it is however not possible to prove convergence. In Section 1.3, we derive some alternative inequalities which are more delicate to handle, but with which it is possible to prove convergence.

The approach in [19] and [9] had originally been purely perturbative, but by an observation of [2], this can be extended to arbitrary $\beta > 0$.

1.2 The skeleton inequalities and boundedness properties

Let $\varepsilon > 0$, $0 \leq t < t + \varepsilon \leq T < \infty$, and set

$$J^{\varepsilon}_{t,T}(\omega) \stackrel{\text{def}}{=} \int_t^{T-\varepsilon} ds_1 \int_{s_1+\varepsilon}^{T} ds_2 \ \delta(\omega_{s_1} - \omega_{s_2})$$

which can be defined as the a.s. limit

$$\lim_{a \downarrow 0} \int_t^{T-\varepsilon} ds_1 \int_{s_1+\varepsilon}^{T} ds_2 \ p_a(\omega_{s_1} - \omega_{s_2}).$$

For $T - t \leq \varepsilon$, we put $J^{\varepsilon}_{t,T}(\omega) = 0$. The existence of this limit can be proved by Fourier techniques (see [64]). We will below perform some formal manipulations with δ-functions, which all can easily been justified (for a fixed ε-gap) by replacing δ by p_a and letting $a \to 0$. All the serious trouble is coming when discussing the $\varepsilon \to 0$ limit, and we will focus on that.

We consider the so called two point functions $\overline{g}^{\varepsilon}_{T,\beta}(x)$ defined to be the density of the measure

$$E_T\left(\exp(-\beta J^{\varepsilon}_{0,T});\ \omega_T \in dx\right)$$

on $\mathbb{R}^d$. It is convenient to write this formally as

$$\overline{g}^{\varepsilon}_{T,\beta}(x) = E_T\left(\exp(-\beta J^{\varepsilon}_{0,T})\delta(\omega_T - x)\right).$$

We write $\overline{g}$ because these quantities have to be slightly modified later on, and we will switch then to g.

Evidently, we have for $0 \leq t \leq T$

$$p_t * \overline{g}^{\varepsilon}_{T-t,\beta} = E_T\left(\exp(-\beta J^{\varepsilon}_{t,T})\delta(\omega_T - x)\right).$$

Setting $t = 0$ gives $\overline{g}_T$ and $t = T$ gives p_T. By the fundamental theorem of calculus, we therefore arrive at

$$\begin{aligned} p_T(x) - \overline{g}^{\varepsilon}_{T,\beta}(x) &= \int_0^T dt \frac{d}{dt}(p_t * \overline{g}^{\varepsilon}_{T-t,\beta})(x) \\ &= \int_0^T dt\ E_T\left(-\beta\left(\frac{d}{dt}J^{\varepsilon}_{t,T}\right)\exp(-\beta J^{\varepsilon}_{t,T})\delta(\omega_T - x)\right). \end{aligned}$$

Now, $\frac{d}{dt}J^{\varepsilon}_{t,T}(\omega) = -\int_{t+\varepsilon}^{T} ds\ \delta(\omega_s - \omega_t)$, if $t \leq T - \varepsilon$, and 0 otherwise, and we therefore get

$$p_t(x) - \overline{g}^{\varepsilon}_{T,\beta}(x) = \beta \int_0^{T-\varepsilon} dt \int_{t+\varepsilon}^{T} ds\, E(\delta(\omega_s - \omega_t)\exp(-\beta J^{\varepsilon}_{t,T})\delta(\omega_t - x)). \tag{1.5}$$

The manipulation may look somewhat cavalier, in particular as the intersection local times are not differentiable in the time limits, but they are easily justified. We will derive some concrete inequalities involving $\overline{g}$. These inequalities are the only objects we are interested in. These inequalities do make sense also when δ is replaced by p_a in which case all the manipulations are easily justified, and we can take the $a \to 0$ limit in the end. We will however stick to the δ notation which is evidently more convenient. We will often drop ε, β in the notations but they should be remembered to be present. On the right hand side of (1.5), we can split the interaction on $[t, T]$ into the self-interactions on $[t, s]$ and $[s, T]$ and the interactions between these intervals:

$$J^{\varepsilon}_{t,T} = J^{\varepsilon}_{t,s} + J^{\varepsilon}_{s,T} + J^{\varepsilon}_{t,s;s,T}, \tag{1.6}$$

where

$$J^{\varepsilon}_{t,s;s,T} = \iint\limits_{\substack{t \le s_1 \le s \le s_2 \le T \\ s_2 - s_1 \ge \varepsilon}} ds_1 ds_2 \ \delta(\omega_{s_1} - \omega_{s_2}). \tag{1.7}$$

In (1.5), there is no interaction inside the interval $[0, t]$, and also none between this interval and the next. However, there is an interaction left between $[t, s]$ and $[s, T]$, which is given by the third summand on the right hand side of (1.6). Without this interaction, the right hand side of (1.5) would just be

$$\int_0^{T-\varepsilon} dt \int_{t+\varepsilon}^{T} ds \int dy \ p_t(y) \overline{g}_{s-t}(0) \overline{g}_{T-s}(x-y). \tag{1.8}$$

(We have dropped β, ε in $\overline{g}$ for notational convenience.)

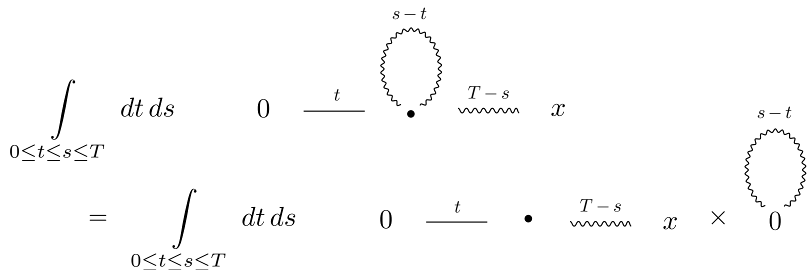
It is convenient to introduce a diagram notation for this and more complicated expressions. The so-called free propagator is

$$0 \ \overset{T}{\text{———}} \ x \ \stackrel{\text{def}}{=} \ p_T(x),$$

and the propagator with interaction is

$$0 \ \overset{T}{\sim\!\sim\!\sim} \ x \ \stackrel{\text{def}}{=} \ \overline{g}_T(x),$$

One should however keep in mind that there is always an ε present: otherwise this interactive propagator is not defined (yet). We also always define $\overline{g}_T(x) = 0$ for $T < \varepsilon$. We can then write the expression (1.8) as



where the $\cdot$ means that we take convolution in x-space (i.e. in $\mathbb{R}^3$).

The main trouble is evidently coming from the presence of the interaction summand of (1.6). This is now handled by some simple inequalities which use the fact that this interaction term is non-negative. We therefore get the two inequalities

$$e^{-\beta J^\varepsilon_{t,s;s,T}} \geq 1 - \beta J^\varepsilon_{t,s;s,T}, \tag{1.9}$$

and

$$e^{-\beta J^\varepsilon_{t,s;s,T}} \leq 1 - \beta J^\varepsilon_{t,s;s,T} + \frac{\beta^2}{2}(J^\varepsilon_{t,s;s,T})^2. \tag{1.10}$$

Implementing the second summand in the right hand side of (1.5) gives a contribution

$$\begin{aligned} &-\beta^2 \int_{A(\varepsilon)} ds_1 ds_2 ds_3 ds_4 \int dy \int dz\, p_{s_1}(y) \\ &\times E\big[e^{-\beta\left(J^\varepsilon_{s_1,s_3}+J^\varepsilon_{s_3,T}\right)}\ \delta(\omega_{s_1}-y)\delta(\omega_{s_{s_2}}-z) \\ &\times \delta(\omega_{s_3}-y)\delta(\omega_{s_4}-z)\delta(\omega_T-x)\big], \end{aligned} \tag{1.11}$$

where $A(\varepsilon) = \{(s_1,s_2,s_3,s_4) :\ 0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T,\ s_3 - s_1 \geq \varepsilon,\ s_4 - s_2 \geq \varepsilon\}$.

It should now be observed that $J^\varepsilon_{s_1,s_3}$ still contains all the interaction on the interval $[s_1,s_3]$ and similarly for $J^\varepsilon_{s_3,T}$. It looks that we have gained nothing as things are becoming more and more complicated. However, dropping the remaining interaction between the intervals $[s_1,s_2]$ and $[s_2,s_3]$, and between $[s_3,s_4]$ and $[s_4,T]$ gives an estimate in the right direction if we use this together with (1.9), simply because

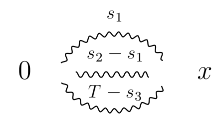
$$J^\varepsilon_{s_1,s_3} \geq J^\varepsilon_{s_1,s_2} + J^\varepsilon_{s_2,s_3},\ J^\varepsilon_{s_3,T} \geq J^\varepsilon_{s_3,s_4} + J^\varepsilon_{s_4,T}.$$

As the remaining propagator will be crucial, we give it a new name:

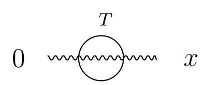
$$\overline{G}^\varepsilon_T(x) = \iint\limits_{\substack{0\le s_1\le s_2\le T\\ s_2\ge\varepsilon, T-s_1\ge\varepsilon}} ds_1 ds_2\ \overline{g}_{s_1}(x)\overline{g}_{s_2-s_1}(x)\overline{g}_{T-s_2}(x). \tag{1.12}$$

In diagram notation, we have

$$\overline{G}^{\varepsilon}_{T}(x) = \iint\limits_{\substack{0\le s_1\le s_2\le T\\ s_2\ge\varepsilon, T-s_1\ge\varepsilon}} ds_1\, ds_2$$



As this propagator is quite crucial, we introduce a new notation for it:

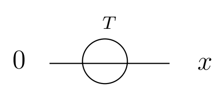


$$\stackrel{\text{def}}{=} \overline{G}^{\varepsilon}_{T}(x)$$

We will also need the corresponding propagator where the $\overline{g}$ are replaced by the free propagator p, but where the ε-restrictions on the integration are kept. This is denoted by

$$P^{\varepsilon}_{T}(x) \stackrel{\text{def}}{=} \iint\limits_{\substack{0\le s_1\le s_2\le T\\ s_2\ge\varepsilon, T-s_1\ge\varepsilon}} ds_1\, ds_2 p_{s_1}(x) p_{s_2-s_1}(x) p_{T-s_2}(x),$$

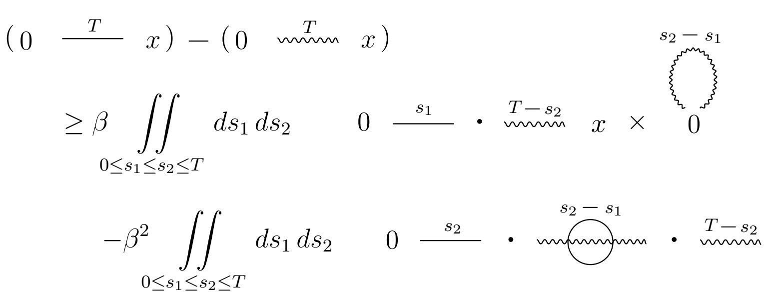
for which we use the diagram notation



$$\stackrel{\text{def}}{=} P^{\varepsilon}_{T}(x)$$

$P^{\varepsilon}_{T}(x)$ is not defined for $x = 0$ as, even with the gap-condition, the integral is divergent. However, for $x \neq 0$, the integrals are perfectly convergent.

In order to recast the first inequality of (1.9), we still have to look at the contribution of 1. Implementing this part into (1.5) just means that we forget about the interaction between $[t, s]$ and $[s, t]$. We therefore get our first basic inequality:

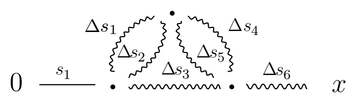


(1.13)

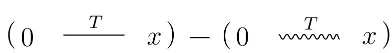
To get an upper bound, we have to expand the interaction between the legs in (1.11) in the same way as before, and we have also to take into account the third summand of the second inequality in (1.10). The reader will convince himself quickly that all the contributions are of the same form, namely

$$\int_{A_3(\varepsilon)} d\underline{s}\ (p_{s_1} * [\overline{g}_{\Delta s_3}((\overline{g}_{\Delta s_1}\overline{g}_{\Delta s_2}) * (\overline{g}_{\Delta s_4}\overline{g}_{\Delta s_5}))] * \overline{g}_{\Delta s_6})(x) \tag{1.14}$$

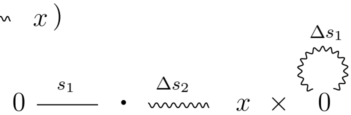
$$= \int_{A_3(\varepsilon)} ds$$



where $A_3(\varepsilon)$ is some subset of $\{\underline{s} = (s_1, s_2, \dots, s_6) : 0 \le s_1 \le s_2 \le \dots \le s_6 \le T\}$ with a number of ε-gap conditions, whose exact form will be no longer of any importance, and $\Delta s_i = s_{i+1} - s_i$ ($s_7 = T$). We will stick to this notation of Δs_i also in cases where the number m of different s_i is not 6, always putting $\Delta s_m = T - s_m$. The above contribution pops up from multiplying out the square of the third summand in (1.10), dropping afterwards all the remaining interactions between the time slots there, and by expanding the interaction between the time slots in (1.11) once. The inequalities evidently all go in the correct direction to yield

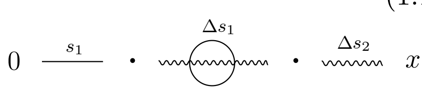


$$\le \beta \int_{0 \le s_1 \le s_2 \le T} d\underline{s}$$

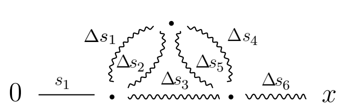


$$(1.15)$$

$$- \beta^2 \int_{0 \le s_1 \le s_2 \le T} d\underline{s}$$



$$+ 4\beta^3 \int_{0 \le s_1 \le \dots \le s_6 \le T} d\underline{s}$$



It is worthwhile to pause and contemplate if anything has been achieved with the inequalities (1.13) and (1.15). A moment's reflection reveals that this is not the case. For instance, the first diagram contains an integration over a "loop" $\overline{g}_s(0)$ over a time $\ge \varepsilon$. If for the moment, we let drop the interaction completely, we have $\int_\varepsilon p_s(0)ds$ which is divergent as $\varepsilon \to 0$. One might think that the interaction could help and $\int_\varepsilon \overline{g}_s(0)ds$ would be convergent, but this is not the case. For similar reasons, the second summand is divergent for $\varepsilon \to 0$ (but actually only marginally). We are more fortunate with the third summand on the right hand side of (1.15). If we drop all the interactions we arrive at

$$\int_{A_3(\varepsilon)} ds\, (p_{s_1} * [p_{\Delta s_3}((p_{\Delta s_1} p_{\Delta s_2}) * (p_{\Delta s_4} p_{\Delta s_5}))] * p_{\Delta s_6})(x),$$

and if we drop also all the gap conditions (this gives an estimate from above), we arrive at

$$\int_{0 \le s_1 \le s_2 \le s_3 \le s_4 \le s_5 \le s_6 \le T} ds \{p_{s_1} * [p_{\Delta s_3}((p_{\Delta s_1} p_{\Delta s_2}) * (p_{\Delta s_4} p_{\Delta s_5}))] * p_{\Delta s_6}\}(x),$$

and it is elementary to check that this is convergent! (For $d = 3$.) Therefore, there might be some hope that the third summand on the r.h.s. of (1.15) is o.k. The way to get also the first two summands right is to modify the definition of $\overline{g}$ slightly by introducing so-called counterterms which are cancelling these divergencies. We will then prove pointwise boundedness and decay properties by an appropriate recursion Ansatz, assuming β small enough.

As these counterterms are supposed to cancel the loop and "three leg" divergency, we define them by the corresponding objects for the free propagator:

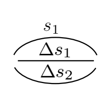
$$\kappa_1(\varepsilon) = \int_{\varepsilon}^{1} ds\, p_s(0) \tag{1.16}$$

and

$$\begin{aligned} \kappa_2(\varepsilon) &= \int_0^1 ds\ \|P_s^{\varepsilon}\|_1 \\ &= \iiint_{\substack{0 \le s_1 \le s_2 \le s_3 \le 1 \\ s_2 \ge \varepsilon, s_3 - s_1 \ge \varepsilon}} ds_1 ds_2 ds_3 \int dx\ p_{s_1}(x) p_{s_2 - s_1}(x) p_{s_3 - s_1}(x), \end{aligned}$$

i.e.

$$\kappa_2(\varepsilon) = \int_{\substack{0 \le s_1 \le s_2 \le s_3 \le 1 \\ s_2 \ge \varepsilon, s_3 - s_1 \ge \varepsilon}} \Bigg\| \ 0$$



$$\bullet \ \Bigg\|_1$$

$\kappa_1(\varepsilon)$ is of course just $2(2\pi)^{-3/2}(\frac{1}{\sqrt{\varepsilon}} - 1)$. κ_2 is slightly more complicated to evaluate. Remark first that

$$\int dx\, p_{t_1}(x) p_{t_2}(x) p_{t_3}(x) = (2\pi)^{-3} [t_1 t_2 + t_1 t_3 + t_2 t_3]^{-3/2},$$

and therefore

$$
\begin{aligned}
\kappa_2(\varepsilon) &= (2\pi)^{-3}\int_0^1 dt_1\int_0^1 dt_3 \int_{(\varepsilon-t_1)\vee(\varepsilon-t_3)\vee 0}^1 dt_2\,[(t_1+t_3)t_2+t_1t_3]^{-3/2}+O\,(1)\\
&= (2\pi)^{-3}\int_0^1 \frac{du}{u}\int_0^u \frac{2dv}{\sqrt{v(u-v)+u\left((\varepsilon-v)\vee(\varepsilon-v+u)\right)_+}}+O\,(1)\\
&= (2\pi)^{-3}\int_{3\varepsilon}^1 \frac{du}{u}\int_\varepsilon^{u-\varepsilon} \frac{2dv}{\sqrt{v(u-v)+u\left((\varepsilon-v)\vee(\varepsilon-v+u)\right)_+}}+O\,(1)\\
&= (2\pi)^{-3}\int_{3\varepsilon}^1 \frac{du}{u}\int_0^u \frac{2dv}{\sqrt{v(u-v)}}+O\,(1)=(2\pi)^{-2}|\log\varepsilon|+O\,(1)\,.
\end{aligned}
$$

We therefore see that this is just barely divergent. The divergence of $\kappa_2(\varepsilon)$ is actually making all the trouble for $d=3$. It should be remarked that $\kappa_1(\varepsilon)$ is essentially just $EJ^{\varepsilon}_{0,1}$ and $\kappa_2(\varepsilon)$ the variance. If the variance stays bounded as $\varepsilon\to 0$, one can apply what in quantum field theory is called vacuum renormalization, i.e. one just replaces J by $J-EJ$ getting something which is convergent. This is the approach of Varadhan for $d=2$ [75]. The renormalized interaction is now just

$$R^{\varepsilon,\beta}_{t,T}=\beta J^{\varepsilon}_{t,T}-\beta(T-t)\kappa_1(\varepsilon)+\beta^2(T-t)\kappa_2(\varepsilon).$$

It is important that the time enters linearly into the counterterms. We put

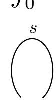
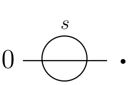
$$g^{\varepsilon}_{T,\beta}(x)\overset{\text{def}}{=}E_T(\exp(-R^{\varepsilon,\beta}_{0,T})\delta(\omega_T-x)).$$

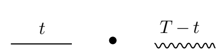
We again apply

$$p_T(x)-g^{\varepsilon}_{T,\beta}(x)=\int_0^T dt\frac{d}{dt}E(\exp(-R^{\varepsilon,\beta}_{t,T})\delta(\omega_T-x)).$$

There are only small changes to our inequalities (1.13) and (1.15). The presence of the counterterms gives only the additional summand

$$(-\beta\kappa_1(\varepsilon)+\beta^2\kappa_2(\varepsilon))\int_0^T dt\; p_t * g_{T-t}(x) \tag{1.17}$$

$= \Big(-\beta\int_\varepsilon^1 ds$  $0 \;+\;\beta^2\int_0^1 ds\;\Big\|\,0$  $\Big\|_1\Big)$

$\times\int_0^T dt\quad 0$  x

in both cases.

We don't change our diagram notations, but from now on, the propagators with interactions always refer to g and not to $\overline{g}$, i.e. to the propagators including the counterterms.

The main pointwise estimate for small $\beta > 0$ is

Proposition 1.3. *There exists $C > 0$ and $\beta_0 > 0$, such that for all $\beta \in [0, \beta_0]$, $T \leq 1$, $x \in \mathbb{R}^3$, $\varepsilon > 0$ one has*

$$|p_T(x) - g^{\varepsilon}_{T,\beta}(x)| \leq C\beta\sqrt{T}p_{2T}(x).$$

The proof of this estimate is by a recursion argument using our basic inequalities (1.13) and (1.15) with the appropriate corrections coming from the counterterms, i.e. (1.17). We set

$$K_0(\varepsilon, \beta) = \sup_{0<T\leq 1} \sup_{x\in\mathbb{R}^3} \frac{|g^{\varepsilon}_{T,\beta}(x) - p_T(x)|}{\sqrt{T}p_{2T}(x)}.$$

Remark first, that $K_0(\varepsilon, \beta)$ is finite for fixed $\varepsilon > 0$, $\beta \geq 0$. In fact, for $T \leq \varepsilon$, the interaction is 0, so

$$g^{\varepsilon}_{T,\beta}(x) = p_T(x) \exp(\beta T \kappa_1(\varepsilon) - \beta^2 T \kappa_2(\varepsilon)),$$

so the sup over $0 < T \leq \varepsilon$ is certainly finite, as $p_T(x)$ decays faster at $|x| \sim \infty$ than $p_{2T}(x)$. For the same reason, the supremum is also finite on $\varepsilon \leq T \leq 1$. $K_0(\varepsilon, \beta)$ looks being the right quantity for Proposition 1.3, but for technical reasons, we have to slightly change it, and we set

$$K(\varepsilon, \beta) = K_0(\varepsilon, \beta) \vee \left| \int_0^1 (p_s(0) - g^{\varepsilon}_{s,\beta}(0))ds \right|.$$

The reason is that in order to estimate K_0, one has to use estimates on $\int\limits_0 (p_s - g_s)ds$. Evidently, this quantity itself cannot be controlled by K_0. This is a slightly awkward point, and for that reason we have to work with K instead of K_0.

The main work for proving Proposition 1.3 is then contained in

Proposition 1.4. *There exists a polynomial $\phi(x)$ with nonnegative coefficients, such that for all $\varepsilon > 0, \beta \in [0, 1]$, one has*

$$K(\varepsilon, \beta) \leq \beta\ \phi(K(\varepsilon, \beta)).$$

The proof of this is a bit lengthy and tedious but essentially rather straightforward. We give details of some parts of the estimates, namely the ones involving the divergent "three leg" diagram. In the next section where we prove convergence, we then focus on the other divergent part. Before we begin with that, we show how Proposition 1.4 implies Proposition 1.3. We still need a further result

Lemma 1.5. *For any fixed $\beta > 0$, the function $(0, 1) \ni \varepsilon \to K(\varepsilon, \beta)$ is continuous.*

This follows well known techniques concerning intersection local times (see [64], and the Appendix of [9]). We will not give a proof here. The Proposition 1.4 and Lemma 1.5 imply Proposition 1.3 in the following way:

Let

$$\varrho(\beta) \stackrel{\text{def}}{=} \inf\{x \geq 0 : x = \beta\phi(x)\}.$$

If β is small enough then we have $\varrho(\beta) \leq c\beta$. We have $K(\varepsilon, \beta) = 0$ for $\varepsilon = 1$, and as $K(\varepsilon, \beta)$ is continuous in $\varepsilon > 0$, it can never cross $\varrho(\beta)$. We therefore get the estimate $K(\varepsilon, \beta) \leq c\beta$ for all $\varepsilon > 0$ if β is small enough. This proves Proposition 1.3.

To come now to the proof of Proposition 1.4, we get, using our inequalities (1.13) and (1.15) with the correction (1.17) from the counterterms:

$$\begin{aligned}
&|p_T(x) - g_T(x)| \\
&\quad = \Big|\big(0 \xrightarrow{\;T\;} x\big) - \big(0 \overset{T}{\rightsquigarrow} x\big)\Big| \\
&\quad \leq \beta \,\Bigg|\int_{0 \leq s_1 \leq s_2 \leq T} d\underline{s}\;\; 0 \xrightarrow{\;s_1\;} \bullet \overset{\Delta s_2}{\rightsquigarrow} x \times \overset{\Delta s_1}{\circlearrowleft}0 \\
&\qquad\qquad - \kappa_1(\varepsilon) \int_0^T ds\;\; 0 \xrightarrow{\;s\;} \bullet \overset{T-s}{\rightsquigarrow} x\Bigg| \qquad (1.18) \\
&\quad + \beta^2 \,\Bigg|\int_{0 \leq s_1 \leq s_2 \leq T} d\underline{s}\;\; 0 \xrightarrow{\;s_1\;} \bullet \overset{\Delta s_1}{\rightsquigarrow\!\bigcirc\!\rightsquigarrow} \bullet \overset{\Delta s_2}{\rightsquigarrow} x \\
&\qquad\qquad - \kappa_2(\varepsilon) \int_0^T ds\;\; 0 \xrightarrow{\;s\;} \bullet \overset{T-s}{\rightsquigarrow} x\Bigg| \\
&\quad + 4\beta^3 \int ds\;\; 0 \xrightarrow{\;s_1\;} \bullet \overset{\Delta s_1,\,\Delta s_2,\,\Delta s_3,\,\Delta s_4,\,\Delta s_5}{\rightsquigarrow} \bullet \overset{\Delta s_6}{\rightsquigarrow} x
\end{aligned}$$

Here we have of course the modified definition

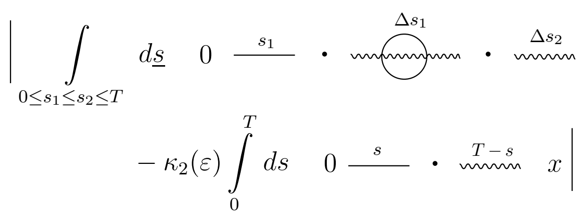
$$0 \overset{s}{\rightsquigarrow\!\bigcirc\!\rightsquigarrow} x \stackrel{\text{def}}{=} G_s(x)$$

but just without the bar, meaning that the appropriate counterterms are included. In the last summand of (1.18), the integration is over time slots for the vertices of the diagram which sum to T. We can drop the various ε-gap restrictions in that contribution, getting an upper bound. In contrast, the other two summands retain the gap restrictions.

It is now fairly obvious, how the necessary cancellations take place. The counterterm $\kappa_1(\varepsilon)$ is defined in such a way that it cancels the loop divergency, and $\kappa_2(\varepsilon)$ is made such that it should cancel the "three leg" divergency. For the third summand, there is no further cancellation needed as it is convergent anyway. Of course, there is the problem that the times to not fit quite nicely, and in the second summand, the three leg divergency has to be "operated out" of the diagram and the remaining two ends "glued together".

We will not present all the estimates in details, but will show how to perform them for the second summand, which is the more delicate one.

Lemma 1.6.

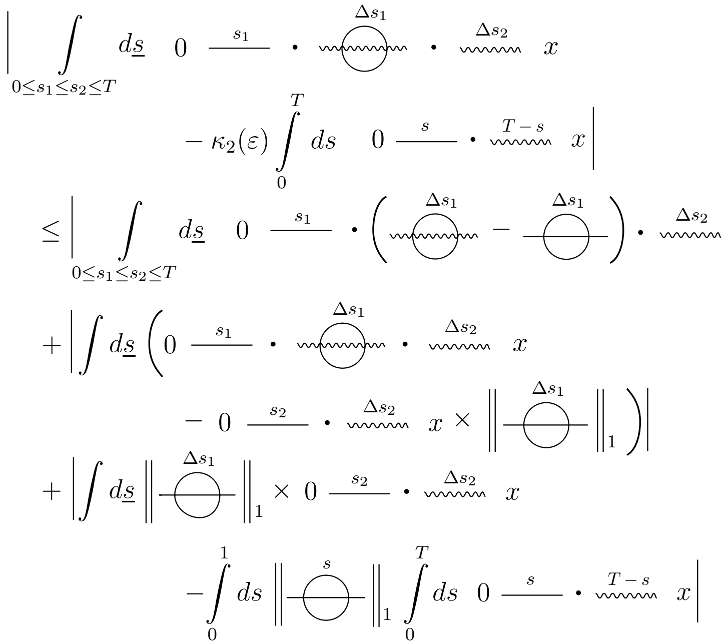


$$\le \phi(K)\,p_{2T}(x)\,T^{3/4}.$$

Notations We use $\phi(x)$ as a generic polynomial with positive coefficients, not necessarily the same at different occurrences. K is always $K(\varepsilon,\beta)$. We also use C as a generic positive constant, also not necessarily the same at different occurrences, which does not depend on ε,β.

Remark 1.7. The estimate is better than necessary for our estimate. The factor with $\sqrt{T}$ appears in the estimate of the first summand on the right hand side of (1.18).

In order to prove the Lemma, we split things into three parts:



(1.19)

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

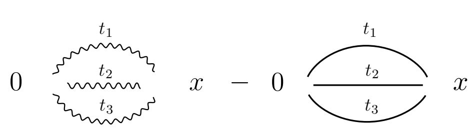
(Remember that G_s and P_s have the gap conditions and particularly are nonzero only if their time length is $\geq \varepsilon$). First remark that for $T \leq 1$, which we always assume, we have

$$|g_T\,(x)| \leq p_T\,(x) + |p_T\,(x) - g_T\,(x)| \leq C\,(1+K)\,p_{2T}\,(x)\,. \tag{1.19}$$

The third summand I_3 is very easy, we begin with that. The only difference between the two contributions inside is that in the first, the integration over Δs_1 is restricted to $\varepsilon \leq \Delta s_1 \leq s_2$. Therefore, using (1.19), we get

$$\begin{aligned} I_3 &= \int_0^T ds\,(p_s * g_{T-s})(x) \int_s^1 \|P_u\|_1 du \\ &\leq C \int_0^T ds (p_s * g_{T-s})(x) |\log(s)| \\ &\leq C\,(1+K)\,p_{2T}\,(x) \int_0^T ds\,|\log(s)| \leq C T^{3/4} p_{2T}(x). \end{aligned} \tag{1.20}$$

Next, we estimate I_1, which is more complicated. We first split



as a sum of expressions of the form $h_{t_1}(x)h_{t_2}(x)h_{t_3}(x)$, where h_t is either $p_t(x)$ or $g_t(x) - p_t(x)$, but where at least one of the h's is the latter. For definiteness, let us look at

$$(g_{t_1}(x) - p_{t_1}(x))p_{t_2}(x)p_{t_3}(x)$$

which we estimate in absolute value by

$$\begin{aligned}&\sqrt{t_1}Kp_{2t_1}(x)p_{t_2}(x)p_{t_3}(x)\\&= C\sqrt{t_1}K[2t_1t_2 + 2t_1t_3 + t_2t_3]^{-3/2}p_\sigma(x),\end{aligned}$$

where

$$\sigma \stackrel{\mathrm{def}}{=} \frac{2t_1t_2t_3}{2t_1t_2 + 2t_1t_3 + t_2t_3}. \tag{1.21}$$

We also replace $g_{\Delta s_2}$ in I_1 by $p_{\Delta s_2}$ and the difference, where we again estimate the latter by $\sqrt{\Delta s_2}Kp_{2\Delta s_2}(x)$. Evidently, the more of the $\sqrt{\Delta s}$ terms we have, the better, so we look what happens if we just have one. Such a contribution to an upper bound of I_1 is

$$\leq CK \int_{\substack{0\leq s_1\leq s_2\leq s_3\leq s_4\leq T\\ s_3-s_1\geq\varepsilon, s_4-s_2\geq\varepsilon}} d\underline{s}\, \frac{\sqrt{\Delta s_1}(p_{s_1+\Delta s_4} * p_\sigma)(x)}{(2\Delta s_1\Delta s_2 + 2\Delta s_1\Delta s_3 + \Delta s_2\Delta s_3)^{3/2}} \tag{1.22}$$

where $d\underline{s} \stackrel{\mathrm{def}}{=} ds_1\, ds_2\, ds_3\, ds_4$, $\Delta s_i \stackrel{\mathrm{def}}{=} s_{i+1} - s_i$ and σ is from (1.21), t_i replaced by Δs_i. Of course, there are also summand with K^2, K^3, K^4 in the estimates (and correspondingly more Δs_i), but these can be estimated similarly.

Let us look at what happens with the expression (1.22). Keeping Δs_1, Δs_2,Δs_3 fixed and integrating over s_1 gives just a factor $t \stackrel{\mathrm{def}}{=} T - (\Delta s_1 + \Delta s_2 + \Delta s_3) = (s_1 + \Delta s_4)$. Furthermore, a simple estimate for σ is $\sigma \leq \sum_i \Delta s_i = T - t$, which yields

$$tp_t(x) \leq t^{-1/2}T^{3/2}p_{T-\sigma}(x).$$

Therefore,

$$\begin{aligned}(1.22) &\leq CKT^{3/2}p_T(x) \int_{t_i\geq 0, \sum t_i<1} \frac{\sqrt{t_1}dt_1dt_2dt_3}{\sqrt{1-\sum t_i}(t_1t_2 + t_1t_3 + t_2t_3)^{3/2}}\\&\leq CKT^{3/2}p_T(x).\end{aligned}$$

Remark that the $\sqrt{t_1}$ is doing the job of making the integral convergent. The other expressions get similar estimates, but we cannot always have $p_T(x)$.

However, $p_{2T}(x)$ is o.k., too, of course. There are also expressions with K^2, K^3, K^4. Altogether, we get

$$I_1 \leq \phi(K) T^{3/2} p_{2T}(x), \tag{1.23}$$

which is much better than required.

There remains to estimate I_2 (which is the only place where we have to increase T to $2T$ in the estimate). Even if we drop all ε restrictions, $P_t^\varepsilon(x)$ is evidently finite for $x \neq 0$, and has L_1-norm $\|P_t^0\|_1 = c/t$. Despite the fact that this is divergent for $t \to 0$, it is nevertheless true that for $t \sim 0$, is $P_t^\varepsilon(x)$ is essentially concentrated close to 0, and therefore there is not much difference between $p_s * P_t^\varepsilon * g_u$ and $(p_s * g_u) \times \|P_t^\varepsilon\|_1$.

If we set

$$\tau(s) \stackrel{\text{def}}{=} [\Delta s_1 \Delta s_2 + \Delta s_1 \Delta s_3 + \Delta s_2 \Delta s_3]^{-3/2},$$

$$\sigma(s) \stackrel{\text{def}}{=} \Delta s_1 \Delta s_2 \Delta s_3 \tau^{2/3},$$

where as usual $\Delta s_i = s_{i+1} - s_i$, then

$$p_{\Delta s_1}(x) p_{\Delta s_2}(x) p_{\Delta s_3}(x) = (2\pi)^{-3} \tau(s) p_{\sigma(s)}(x).$$

Therefore

$$I_2 \leq C \int d\underline{s}\, \tau(s) (|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x),$$

where the integration is over $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T$, and where we have dropped the ε-gap restrictions. We split the above integral in the part, where $s_1 + \sigma \leq s_4/2$ and the complement of this. On $\{s_1 + \sigma \leq s_4/2\}$ there is actually no cancellation needed and we estimate $|p_{s_1+\sigma} - p_{s_4}| \leq p_{s_1+\sigma} + p_{s_4}$, and of course, we also estimate

$$g_{T-s_4} \leq p_{T-s_4} + K\sqrt{T - s_4} p_{2T-2s_4} \leq \phi(K) p_{2T-2s_4}.$$

Therefore,

$$\begin{aligned}
&\int_{\{s_1+\sigma<s_4/2\}} d\underline{s}\, \tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x) \\
&\qquad \leq \phi(K) \int_{\{s_1<s_4/2\}} d\underline{s}\, \tau(s)(p_{2T-s_4} + p_{2T-2s_4+s_1+\sigma})(x) \\
&\qquad \leq \phi(K) \int_{\{s_1<s_4/2\}} d\underline{s}\, \tau(s) p_{2T}(x) \\
&\qquad\quad + \phi(K) \int_{\{s_1<s_4/2, s_4>T/2\}} d\underline{s}\, \tau(s) p_{2T-2s_4+s_1+\sigma}(x).
\end{aligned}$$

The integration of τ over s_2, s_3 for given s_1, s_4 is just $\|P^0_{s_4-s_1}\|_1 = c|s_4-s_1|^{-1}$, and so the first summand is $\phi(K)T\,p_{2T}(x)$. As for the second, it is

$$\leq \phi(K) \int\limits_{\{s_1<s_4/2, s_4>T/2\}} d\underline{s} \qquad \tau(s)(T - s_4 + s_1)^{-3/2}\mathrm{e}^{-|x|^2/4T}$$

$$\leq \phi(K)\mathrm{e}^{-|x|^2/4T} \int_0^{3T/4} \frac{du}{\sqrt{u}(T-u)} \leq \phi(K)Tp_{2T}(x).$$

Altogether we get

$$\int\limits_{\{s_1+\sigma<s_4/2\}} d\underline{s}\,\tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x) \leq \phi(K)Tp_{2T}(x).$$

It remains to estimate the integral over $\{s_1 + \sigma(s) \geq s_4/2\}$. In that case it would be disastrous to take the absolute value inside. Instead we use the elementary estimate

$$|p_u(x) - p_v(x)| \leq c|u-v|v^{-1}p_{2v}(x),$$

when $v/2 \leq u \leq v$. Using this, we get

$$\int\limits_{\{s_1+\sigma\geq s_4/2\}} d\underline{s}\,\tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x)$$
$$\leq \int d\underline{s}\ \tau(s)(s_4 - s_1 - \sigma)\frac{1}{s_4}p_{2s_4} * g_{T-s_4}(x)$$
$$\leq \phi(K)p_{2T}(x)\int d\underline{s}\,\tau(s)(s_4 - s_1)\frac{1}{s_4} = \phi(K)Tp_{2T}(x),$$

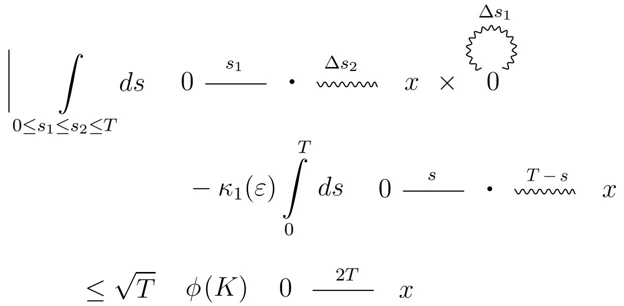
the integrals of course all restricted to $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T$.

Altogether, we have

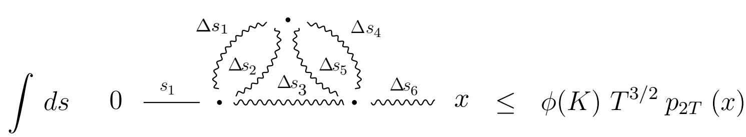
$$I_2 \leq \phi(K)Tp_{2T}(x). \tag{1.24}$$

The estimates (1.23), (1.24) and (1.20) prove Lemma 1.6.

In order to prove Proposition 1.4, there are still the other contributions in (1.18) to estimate, but we will not give the details here, and refer to [9]. There, the following inequalities are proved:



The estimates for this are somewhat easier than for the ,,three leg" divergency, because the loop diagram splits off the rest immediately, and one has only to take care of the adjustments of the times. On the other hand, the divergency is more serious than in second summand of (1.18). In the course to prove the above estimate one needs K at one place and not just K_0. There then remains to estimate the last diagram, where counterterms are appearing:



The crucial observation is that the corresponding diagram with free propagators is convergent, which is a somewhat tedious but elementary calculation, and then one estimates the above diagram by replacing successively the $g's$ by the $p's$, and one catches only additional factors $\phi(K)$.

Combining Lemma 1.6 with the above estimates, one therefore gets

$$K_0(\varepsilon,\beta) \le \beta\phi(K\,(\varepsilon,\beta)). \tag{1.25}$$

It then still remains to estimate

$$\left|\int_0^1 (p_T(0) - g_T(0))dT\right| \le \beta\phi(K). \tag{1.26}$$

As there is a slight slip in the argument in [9], we give the proof here. We use the same expansion which underlies the estimate (1.18). As the second and third summand on the right hand side of (1.18) are at most $\beta^2\phi(K)T^{3/4}p_{2T}(x)$, and $T^{3/4}p_{2T}(0)$ is integrable at 0, it suffices to prove

$$\Bigg| \int_0^1 dT \int\limits_{0 \le s_1 \le s_2 \le T} d\underline{s} \quad p_{s_1} * g_{\Delta s_2}(0) g_{\Delta s_1}(0)$$
$$- \int_0^1 dT \int_0^T ds \; p_s * g_{\Delta s}(0) \int_\varepsilon^T p_s(0) ds \Bigg| \le \phi(K).$$

The left hand side of this expression is

$$\left| \int\limits_{\substack{u+v+t \le 1 \\ v \ge \varepsilon}} du\, dv\, dt (p_u * g_t)(0) g_v(0) - \int\limits_{u+t \le 1} du\, dt (p_u * g_t)(0) \int_\varepsilon^1 dv p_v(0) \right|$$
$$\le \left| \int\limits_{u+t \le 1-\varepsilon} du\, dt (p_u * g_t)(0) \int\limits_\varepsilon^{1-u-t} dv (p_v - g_v)(0) \right|$$
$$+ \left| \int\limits_{u+t \le 1} dudt (p_u * g_v)(0) \int\limits_{1-u-t}^1 p_v(0) dv \right|$$
$$\le K \left\{ \int\limits_{u+t \le 1} du\, dt (p_u * g_t)(0) \right.$$
$$\left. + \int\limits_{u+t \le 1} du\, dt (p_u * g_t)(0) \int\limits_{1-u-t}^1 dv |p_v(0) - g_v(0)| \right\}$$
$$+ C \int\limits_{u+t \le 1} dudt (p_u * g_t)(0) (1-u-t)^{-1/2}.$$

We now estimate g_t by $p_t + \phi(K) p_{2t} \le \phi(K) p_{2t}$ and get that the above expression is

$$\le \phi(K) \left\{ \int\limits_{u+t \le 1} du\, dt (u+2t)^{-3/2} (1-u-t)^{-1/2} \right\} = \phi(K).$$

(1.26) is therefore proved which implies now Proposition 1.4. We already have seen that this implies Proposition 1.3.

We now show that the relevant information contained in Proposition 1.3 can be boosted by a rescaling argument due to Albeverio and Zhou [3] to cover any $\beta > 0$.

Proposition 1.8. *Let $\beta \geq 0, T \leq 1$. Then there exist constants $c_1(\beta)$, $\dots, c_4(\beta) > 0$ with*
a) $g^{\varepsilon}_{T,\beta}(x) \leq c_1(\beta) p_{c_2(\beta)T}(x), \quad \forall \varepsilon > 0.$
b) $c_3(\beta) \leq \|g^{\varepsilon}_{T,\beta}\|_1 \leq c_4(\beta)$.

The basis is the following simple rescaling property

Lemma 1.9. *For all $T, \beta, \varepsilon > 0$*

$$g^{\varepsilon}_{T/2,\beta}(x) = g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}x)\exp(a(\varepsilon,\beta,T)),$$

where $\sup_{\varepsilon>0,T\leq 1} |a(\varepsilon,\beta,T)| < \infty$ *for all* $\beta > 0$.

Proof.

$$g^{\varepsilon}_{T/2,\beta}(x) = E\left(\exp\left[-\beta J^{\varepsilon}_{0,T/2}(\omega) + \frac{\beta T}{2}\kappa_1(\varepsilon) - \frac{\beta^2 T}{2}\kappa_2(\varepsilon)\right]\delta(\omega_{T/2} - x)\right)$$
$$= 2^{3/2} E\left(\exp\left[-\frac{\beta}{\sqrt{2}} J^{2\varepsilon}_{0,T}(\omega) + \frac{\beta T}{\sqrt{2}}\frac{\kappa_1(\varepsilon)}{\sqrt{2}} - \left(\frac{\beta}{\sqrt{2}}\right)^2 T\kappa_2(\varepsilon)\right]\delta(\omega_T - \sqrt{2}x)\right),$$

$$\frac{1}{\sqrt{2}}\kappa_1(\varepsilon) = \frac{1}{\sqrt{2}}\frac{2}{(2\pi)^{3/2}}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right) = \kappa_1(2\varepsilon) + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{2}{(2\pi)^{3/2}},$$
$$\kappa_2(\varepsilon) = \kappa_2(2\varepsilon) + 0(1).$$

Implementing this, we get the conclusion.

Proof of Proposition 1.8 a). From Proposition 1.3 we already know that there exists $\beta_0 > 0$ such that the statement is true for $\beta \leq \beta_0$. We now prove that if the statement is correct for $\beta \leq \hat{\beta}$, then it is correct for $\beta \leq \sqrt{2}\hat{\beta}$.

Let $\beta \leq \sqrt{2}\hat{\beta}, T \leq 1, \varepsilon > 0$. Then

$$\begin{aligned} g^{\varepsilon}_{T,\beta}(x) &= E\left(\exp[-R^{\varepsilon,\beta}_{0,T}]\delta(\omega_T - x)\right) \\ &\leq E\left(\exp[-R^{\varepsilon,\beta}_{0,T/2} - R^{\varepsilon,\beta}_{T/2,T}]\delta(\omega_T - x)\right) \\ &= (g^{\varepsilon}_{T/2,\beta} * g^{\varepsilon}_{T/2,\beta})(x) \\ &\leq e^{2a(\varepsilon,\beta,T)}(g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}\cdot) * g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}\cdot))(x) \\ &\leq 2^{3/2} e^{a(\varepsilon,\beta,T)} c_1 p_{2c_2T}(\sqrt{2}x), \end{aligned}$$

where $c_i = c_i(\beta/\sqrt{2})$. As $p_a(\sqrt{2}x) \leq Cp_a(x)$, this proves the claim.

Proof of Proposition 1.8 b). The upper bound follows from part a), so it remains to prove the lower bound. First remark that from Lemma 1.9 we get

$$\|g^{\varepsilon}_{T/2,\beta}\|_1 \geq c(\beta)\|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1. \tag{1.27}$$

We again use "induction" on β. Assume that the lower bound in Proposition 1.8 b) is correct for $\beta \leq \hat{\beta}$, and assume $\beta \leq \sqrt{2}\hat{\beta}$.

Let $P_{(2)}$ (with corresponding expectation $E_{(2)}$) be the product measure of two independent Brownian motions of length $T/2$. If ω_1, ω_2 are two paths, we write

$$J^{\varepsilon}(\omega_1,\omega_2) = \int_0^{T/2} ds \int_0^{T/2} dt 1_{s+t\geq\varepsilon}\, \delta(\omega_{1,s} - \omega_{2,t}).$$

Then

$$\|g^{\varepsilon}_{T,\beta}\|_1 = E^{(2)} \exp\left[-R^{\varepsilon,\beta}_{0,T/2}(\omega_1) - R^{\varepsilon,\beta}_{0,T/2}(\omega_2) - \beta J^{\varepsilon}(\omega_1,\omega_2)\right].$$

Let $\hat{P}^{\varepsilon,\beta}$ be the polymer measure (with gap ε) on paths of length $T/2$, and $\hat{P}^{\varepsilon,\beta}_{(2)}$ be the corresponding product measure. Then

$$\begin{aligned}\|g^{\varepsilon}_{T,\beta}\|_1 &= \|g^{\varepsilon}_{T/2,\beta}\|_1^2 \hat{E}^{\varepsilon,\beta}_{(2)} \exp(-\beta J^{\varepsilon}) \\ &\geq c(\beta)^2 \|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1^2 \exp(-\beta \hat{E}^{\varepsilon,\beta}_{(2)} J^{\varepsilon}),\end{aligned}$$

by (1.27). By the induction assumption, we have $\|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1 \geq c_1(\beta)$. In order to prove the result, we only have to estimate

$$\begin{aligned}\hat{E}^{\varepsilon,\beta}_{(2)} J^{\varepsilon} = \frac{1}{\|g^{\varepsilon}_{T/2,\beta}\|_1^2} &\iint_{\substack{s+t\geq\varepsilon \\ 0\leq s,t\leq T/2}} ds\, dt\, E_{(2)} \\ &\cdot \left\{ e^{-R^{\varepsilon,\beta}_{0,T/2}(\omega_1)} e^{-R^{\varepsilon,\beta}_{0,T/2}(\omega_2)} \delta(\omega_{1,s} - \omega_{2,t}) \right\}\end{aligned}$$

from above, and we again estimate $\|g^{\varepsilon,\beta}_{T/2}\|_1^2$ from below with (1.27) and the induction assumption.

$$\begin{aligned}&E_{(2)}\left\{\exp(-R^{\varepsilon,\beta}_{0,T/2}(\omega_1) - R^{\varepsilon,\beta}_{0,T/2}(\omega_2))\delta(\omega_{1,s} - \omega_{2,t})\right\} \\ &\leq \int dx g^{\varepsilon}_{s,\beta}(x) g^{\varepsilon}_{t,\beta}(x) \|g^{\varepsilon}_{T/2-s,\beta}\|_1 \|g^{\varepsilon}_{T/2-t,\beta}\|_1 \\ &\leq c(\beta) \int dx p_{c_1(\beta)s}(x) p_{c_1(\beta)t}(x) \\ &\leq c(\beta) p_{c_1(t)(s+t)}(0) = c_2(\beta)(s+t)^{-3/2}.\end{aligned}$$

Integrating over s, t gives the desired claim.

We can already derive an important conclusion

Proposition 1.10. *For all $\beta > 0$ the family of measures*

$$\{\hat{P}^{\varepsilon}_{1,\beta}\}_{\varepsilon>0}$$

is tight.

Proof. The counterterms play of course no rôle for the measures. So

$$\hat{P}^{\varepsilon}_{1,\beta}(d\omega) = \exp(-R^{\varepsilon,\beta}_{0,1}(\omega))P_1(d\omega)/\|g^{\varepsilon}_{1,\beta}\|_1.$$

Therefore for $0 \le t < t+s \le 1$ by Proposition 1.8

$$\begin{aligned}
&\int |\omega_t - \omega_{t+s}|^4 \hat{P}^{\varepsilon}_{1,\beta}(d\omega) \\
&\le C(\beta) \int |\omega_t - \omega_{t+s}|^4 \exp(-R^{\beta,\varepsilon}_{0,1}(\omega))P_1(d\omega) \\
&\le C(\beta) \int |\omega_t - \omega_{t+s}|^4 \exp(-R^{\beta,\varepsilon}_{0,t} - R^{\beta,\varepsilon}_{t,t+s} - R^{\beta,\varepsilon}_{t+1,1})P_1(d\omega) \\
&= C(\beta) \int dx\, dy\, dz\, g_t(x) g_s(y-x) g_{1-t-s}(z-y)|x-y|^4 \\
&\le C(\beta)|t-s|^2.
\end{aligned}$$

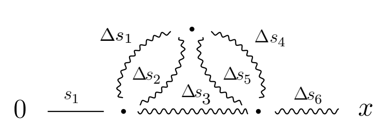
The tightness follows now by standard criteria.

The above proposition of course implies that the measures $\hat{P}^{\varepsilon}_{1,\beta}$ have convergent subsequences as $\varepsilon \to 0$. In the next section, we prove that the $\lim_{\varepsilon\to 0} \hat{P}^{\varepsilon}_{1,\beta}$ exists.

1.3 The convergence of $P^{\varepsilon}_{T,\beta}$, $\varepsilon \to 0$

It is evident that the inequalities presented in Section 1.2 are not able to prove convergence. The reason simply is that the difference of the upper and lower bounds deviate by the contribution

$$\int ds$$



which does not go to 0 as $\varepsilon \to 0$, but only stays finite. We would be much better off, if one of the integrations involved would be only over an interval which becomes small with ε. The idea to achieve something of this type is to differentiate with respect to the gap width $\varepsilon > 0$. To do this, the gap regularization is evidently much better suited than e.g. a lattice regularization. As mentioned in the introduction to this chapter, it can also be proved that the lattice regularization measure converges to the same limit. However, there are considerable additional difficulties popping up and we will not go into that. It should also be remarked that the inequalities we will get by differentiating with respect to $\varepsilon > 0$ are somewhat more delicate to handle for reasons which will become clear. We will heavily rely on the boundedness (and tightness) properties already obtained in order to estimate these diagrams.

Let $\psi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be bounded and smooth, and for $0 \le s < t \le 1$ define $\Psi_{s,t} : \Omega \to \mathbb{R}$ by

$$\Psi_{s,t}(\omega) = \exp\left[\int_s^t \psi(u, \omega_u) du\right].$$

The functions $\Psi = \Psi_{0,1} : \Omega \to \mathbb{R}$ will be convenient for us. They form a convergence determining class, i.e. if we prove that

$$\lim_{\varepsilon \to 0} \int \Psi d\hat{P}^{\varepsilon}_{T,\beta} \tag{1.28}$$

exists (for suitable T, β), then we have proved convergence of the measures, given of course the tightness which is already proved. We fix $T = 1$. Given the estimates in Proposition 1.8, we prove the convergence of the expression (1.28): Let

$$\varrho_\psi(\varepsilon) = \int \Psi \exp(-R^{\beta,\varepsilon}_{0,1}) dP_1.$$

Proposition 1.11. *For any bounded function ψ and all $\beta > 0$ there exists an integrable function $i : (0,\infty) \to (0,\infty)$ such that for any $\varepsilon_2 > \varepsilon_1$*

$$\varrho_\psi(\varepsilon_2) - \varrho_\psi(\varepsilon_1) \ge -\int_{\varepsilon_1}^{\varepsilon_2} i(\varepsilon) d\varepsilon.$$

The bound together with the bounds in Proposition 1.8 immediately prove Theorem 1.1 (for $d = 3$). Indeed as the $\varrho_\psi(\varepsilon)$ stay bounded by Proposition 1.8, Proposition 1.11 implies that $\lim_{\varepsilon \to 0} \varrho_\psi(\varepsilon)$ exists. This together with the tightness proved in Proposition 1.10 proves the convergence of the measures.

We fix now ψ bounded and smooth (with bounded derivatives of all desired order, say) and we write just $\varrho(\varepsilon)$. First, we simply write

$$\varrho(\varepsilon_2) - \varrho(\varepsilon_1) = \int_{\varepsilon_1}^{\varepsilon_2} \frac{d\varrho}{d\varepsilon} d\varepsilon.$$

We actually do not want to prove that $\frac{d\varrho}{d\varepsilon}$ exists. This can be circumvented in the same way as in Section 1.2: We replace all δ function by $p_a's$, derive the necessary inequalities and finally let go $a \to 0$ in the end. This is evidently somewhat cumbersome to write down, so we pretend that we can work directly with the δ function. Differentiating gives

$$\frac{d}{d\varepsilon}\varrho(\varepsilon) = \int_0^{1-\varepsilon} dsE(e^{-R_{0,1}^{\beta,\varepsilon}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) + (\beta\kappa_1'(\varepsilon) - \beta^2\kappa_2'(\varepsilon))\varrho(\varepsilon) \tag{1.29}$$

The crucial point is now as follows. $R_{0,1}^{\beta,\varepsilon}$ of course still contains all the interactions, and we somehow want to expand that out like in the previous section. Especially, we want to expand out the interactions between the interval $[s, s+\varepsilon]$ and its complement. This will lead to contributions which cancel the nonintegrability of the counterterms. The delicacy is coming from the fact that we are *not* allowed to expand the interaction of the time before s and after $s+\varepsilon$ out in any way. Although these contributions are finite, they would, if expanded by Taylor only once or twice, lead to a destruction of all the cancellations. It is therefore better not to expand the interaction crossing the ,,loop interval" $[s, s+\varepsilon]$. We therefore have to control the necessary cancellations in the presence of the interactions of the time before s and after $s+\varepsilon$. Let

$$R_{0,1}^{\beta,\varepsilon} = \tilde{R}_{0,1}^{s,\beta,\varepsilon} + \beta J^{\varepsilon}_{0,s;s,s+\varepsilon} + \beta J^{\varepsilon}_{s,s+\varepsilon;s+\varepsilon,1}, \tag{1.30}$$

where

$$\tilde{R}_{0,1}^{s,\beta,\varepsilon} = R_{0,s}^{\beta,\varepsilon} + R_{s+\varepsilon,1}^{\beta,\varepsilon} + \beta J_{0,s;s+\varepsilon,1} - \beta\varepsilon\kappa_1(\varepsilon) + \beta\varepsilon\kappa_2(\varepsilon). \tag{1.31}$$

As remarked above, the presence of the $J_{0,s;s+\varepsilon,1}$-summand in (1.31) is making a lot of trouble. Of course, we would like to argue that the term obtained when dropping the two last summands on the r.h.s. of (1.30) is cancelling with $\kappa_1'(\varepsilon)$, and expanding these contributions once is cancelling with $\kappa_2'(\varepsilon)$. However, this will be a cancellation of divergent terms (as $\varepsilon \to 0$) and as in Section 1.2, some surgery will be needed to operate the divergency out. The crucial point is that we do not want to expand out any interaction unless it is an interaction connecting an ε-piece to something else. In this way we get estimates which after the cancellation of the divergencies become controllable for $\varepsilon \to 0$.

We give some details for the first part where the contribution coming from $\tilde{R}$ cancels with κ_1'. We then give a sketch how the rest is done the cancellations with κ_2' are done.

We use i as a generic function $(0,\infty) \to (0,\infty)$ which is integrable near 0, not necessarily the same at different occurrences.

Proposition 1.12.

$$\int_0^{1-\varepsilon} dsE(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) + \kappa_1'(\varepsilon)\varrho(\varepsilon) \geq -i(\varepsilon).$$

(We usually drop $\beta, \varepsilon's$ at places where they obviously have to be, e.g. in $\tilde{R}$.)

Proof. We set

$$Y_s = Y^{\varepsilon}_s = \int\limits_{\substack{u \leq s \leq v \leq 1-\varepsilon \\ v-u \leq \varepsilon}} \delta(\omega_u - \omega_v)dudv.$$

Then

$$E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) = E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi_{0,s}\Psi_{s+\varepsilon,1}(1 + O(\varepsilon))).$$

The $1+O(\varepsilon)$ is just the $\Psi_{s,s+\varepsilon}$. There is evidently no interaction inside $[s, s+\varepsilon]$ because of the gap, and in $\tilde{R}$ we have left out the interaction of the "loop" with the rest. We take separately the expectation over $\delta(\omega_s - \omega_{s+\varepsilon})$ which is just $p_\varepsilon(0) = -\kappa_1'(\varepsilon)$, and "glue" the second half of the path to the first, but then the interactions do no longer quite fit, because we no longer have any gap condition between the path before and after s, after having cut the loop out. To restore this, we have to correct by Y_s: Using (1.31) we get

$$\begin{aligned}
&E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) \\
&= p_\varepsilon(0)E(e^{-R_{0,1-\varepsilon}-\beta Y_s}\Psi e^{\beta\varepsilon\kappa_1(\varepsilon)}e^{-\beta^2\varepsilon\kappa_2(\varepsilon)}(1 + O(\varepsilon)) \\
&= p_\varepsilon(0)E(e^{-R_{0,1-\varepsilon}-\beta Y_s}\Psi(1 + \beta\varepsilon\kappa_1(\varepsilon))(1 + O(\varepsilon|\log\varepsilon|)).
\end{aligned}$$

There is, of course, also an adjustment of Ψ by the cutting out of the loop interval, but this give only a contribution $1 + O(\varepsilon)$. It is evident from the considerations in Section 1.2 that $E\exp(-R^s_{0,1-\varepsilon} - \beta Y^{\varepsilon}_s)$ stays bounded (as $\varepsilon \to 0$), and so we can neglect the $0(\varepsilon|\log\varepsilon|)$ contribution as $p_\varepsilon(0)\varepsilon|\log\varepsilon|$ is integrable at 0. However $p_\varepsilon(0)\varepsilon\kappa_1(\varepsilon)$ is not integrable, a fact with which we are pleased as it will cancel the contribution coming from Y_s. As $Y_s \geq 0$, we get

$$e^{-\beta Y_s} \geq 1 - \beta Y_s,$$

and therefore

$$\begin{aligned}
&\int_0^{1-\varepsilon} ds\ E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) \\
&\geq p_\varepsilon(0)(1 + \beta\varepsilon\kappa_1((\varepsilon))\int_0^{1-\varepsilon} ds\left\{E(e^{-R_{0,1-\varepsilon}}\Psi) - \beta E(Y_s e^{-R_{0,1-\varepsilon}}\Psi)\right\} - i(\varepsilon).
\end{aligned}$$

It looks obvious that

$$E(e^{-R_{0,1-\varepsilon}}\Psi) = E(e^{-R_{0,1}}\Psi) + O(\varepsilon) = \varrho(\varepsilon) + O(\varepsilon),$$

but I don't know how to prove this. We would need something like a bound for

$$\frac{d}{dv}E(e^{-R_{0,v}}\Psi).$$

That is close to what we have done in Section 1.2, but there the integration over v was important. (That we differentiate here with respect to the upper boundary in contrast to the lower is of course of no relevance.) However, one can squeeze out of the arguments in Section 1.2 a slightly worse bound which is good enough for our purpose:

Lemma 1.13. *There exists* $\delta > 0$ *such that*

$$|E(e^{-R_{0,1}}\Psi) - E(e^{-R_{0,1-\varepsilon}}\Psi)| \leq C\varepsilon^{1/2+\delta}.$$

We will not give a proof here as it is essentially a repetition of some of the steps of Section 1.2 (see p. 96 of [9]).

End of proof of Proposition 1.12. With the help of Lemma 1.13, the proof is now easily finished: We have

$$\begin{aligned} E(Y_s e^{-R_{0,1-\varepsilon}}\Psi) &= \int\limits_{\substack{u\leq s\leq v\\ v-u\leq\varepsilon}} dudv\ E(\delta(\omega_u - \omega_v)e^{R_{0,1-\varepsilon}}\Psi) \\ &\leq \int\limits_{\substack{u\leq s\leq v\\ v-u\leq\varepsilon}} p_{v-u}(0)E(e^{-R_{0,1-\varepsilon-(v-u)}}\Psi) + 0(\varepsilon^{3/2}) \end{aligned}$$

where we have just dropped the interactions between the interval $[u, v]$ and the rest, and the "less than ε" interaction after readjusting time. This increases the expression. The readjustment of Ψ gives only

$$O\left(\varepsilon\right)\int_{\substack{0\leq u\leq s\leq v\\ v-u\leq\varepsilon}} dv\, du\, p_{v-u}\left(0\right) = O\left(\varepsilon^{3/2}\right).$$

which we can incorporate into $i(\varepsilon)$. By Lemma 1.13, we can replace $R_{0,1-\varepsilon-(v-u)}$ by $R_{0,1}$, making an error which again can be incorporated into $i(\varepsilon)$. Therefore, we get

$$\begin{aligned}
&\int_0^{1-\varepsilon} ds\ E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s-\varepsilon})\Psi) \\
&\geq p_\varepsilon(0)(1+\beta\varepsilon\kappa_1(\varepsilon))\varrho(\varepsilon)\left(1 - \beta\int_0^{1-\varepsilon} ds \int\limits_{\substack{u\leq s\leq v\\ v-u\leq\varepsilon}} p_{v-u}(0)dudv\right) - i(\varepsilon) \\
&= p_\varepsilon(0)\varrho(\varepsilon) - i(\varepsilon),
\end{aligned}$$

as $\varepsilon\kappa_1(\varepsilon) = \int_0^{1-\varepsilon} ds \int\limits_{\substack{u\leq s\leq v\\ v-u\leq\varepsilon}} p_{v-u}(0)du\,dv + O(\varepsilon)$, and as $p_\varepsilon(0) = -\kappa_1'(\varepsilon)$, this proves Proposition 1.12.

From Proposition 1.12, we get

$$\begin{aligned}
\frac{d}{d\varepsilon}\rho(\varepsilon) \geq -\beta^2 \int_0^{1-\varepsilon} ds\, E\left(\mathrm{e}^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\left[J_{0,s;s,s+\varepsilon} + J_{s,s+\varepsilon;s+\varepsilon,1}\right]\Psi\right) \\
\times (1 + O(1/\sqrt{\varepsilon})) - \lambda^2 \frac{d}{d\varepsilon}\kappa_2(\varepsilon) - i(\varepsilon).
\end{aligned} \tag{1.32}$$

In order to finish the proof of Proposition 1.11, it therefore only remains to show that there is some cancellation between the first and the second summand on the right hand side of the above inequality, which leads to something integrable in ε. $\frac{d}{d\varepsilon}\kappa_2(\varepsilon)$ is of order $1/\varepsilon$, so it is clear that not much cancellation is needed. This helps very much, and allows for application of relatively crude estimates. On the other hand, it is also clear that the cancellation is here somewhat more subtle than the one in Proposition 1.12, because the three leg diagram is more delicate to handle than the loop one.

I will not give the details here of the estimates, as it is a bit repetitive of what had been done in Proposition 1.11 (and in the last Section). Here a short outline: One of the problems is of course that $J_{0,s;s,s+\varepsilon}$ and $J_{s,s+\varepsilon;s+\varepsilon,1}$ contain interactions which go outside the interval $[s, s+\varepsilon]$, so they come into conflict with $\tilde{R}^s_{0,1}$. As remarked at the beginning of this section, it is not possible to cancel or expand the interaction inside $\tilde{R}^s_{0,1}$ which ties the part before and that after s. However, it turns out that we can essentially neglect the interactions inside $\tilde{R}^s_{0,1}$ which come into conflict with the above J-terms. What helps here a lot is the fact that the divergency is only logarithmic, and one can work with somewhat crude estimates. What one does is to choose some parameter $0 < \gamma < 1$, whose value is not very important, and cut out from $\tilde{R}^s_{0,1}$ all the interactions with the intervals $[s-\varepsilon^\gamma, s]$ and $[s+\varepsilon, s+\varepsilon+\varepsilon^\gamma]$. However, we retain (this is crucial) the interaction between $[0, s-\varepsilon^\gamma]$ and $[s+\varepsilon+\varepsilon^\gamma, 1]$. This surgery cost an error which can be incorporated into $i(\varepsilon)$. This is essentially an argument like the one involving the Lemma 1.13 above.

Likewise, we drop inside $J_{0,s;s,s+\varepsilon}$ the interaction between $[0, s-\varepsilon^\gamma]$ and $[s, s+\varepsilon]$, and similarly for $J_{s,s+\varepsilon;s+\varepsilon,1}$. In this way, we keep the interactions inside (1.32) separated, and we can now operate the divergency out, cancelling with the derivative of κ_2. There arise now the same problems we had encountered in the last section, namely, that in contrast with the situation with the loop diagram, one has to "glue" the two loose ends together, after taking out the three-leg diagram, but this can essentially be handled in the same way as we did it in details in the last section. One then still has to restore the interaction with the now "void" interval $[s - \varepsilon^\gamma, s + \varepsilon + \varepsilon^\gamma]$, and one has to show that this gives again an error which can be incorporated into $i(\varepsilon)$.

The whole procedure is a bit messy and needs some care, but it should be fairly evident that with the tricks already developed, this can be done, and (1.32) can be proved in this way, leading then to Proposition 1.11. For further details, see the [9].

It should be emphasized that the above considerations do not depend on having $\beta > 0$ small. The argument are valid as long as the estimates of Section 1.2 are true, that is, according to Proposition 1.8, for all $\beta > 0$. So Proposition 1.11 follows for all $\beta > 0$.

I would like to finish this chapter with two

Remarks:

- A shortcoming of the above argument is that we have used at various places that Ψ is a smooth function. In fact, I don't have a proof that $\lim_{\varepsilon\to 0} g^{\varepsilon}_{T,\beta}(x)$ exists, although there is no reasonable doubt that it is true. To prove this would need refinements of the arguments at several places, and would probably be quite delicate.
- Finally some comments about the direct x-space method developed here: Various considerations would become, of course, simpler in Fourier-space, and applying Laplace transforms in time. It is, however, somewhat delicate to implement the monotonicity properties coming from the positivity of the interaction into properties of the Fourier-transforms. Also the diagrams are most easily estimated if one has good pointwise estimates of the two-point functions. Also taking Laplace transforms in time leads in the end to the problem to invert these in order to get results for fixed time. This inversion is notoriously difficult. So, in the end, I believe that working directly in x-space and at fixed time is giving better results.
  Another example where direct x-space methods have been developed is the recent new method to prove diffusivity for weakly self-avoiding random walks for $d \geq 5$ [18] which leads to somewhat stronger results obtained in a more direct and transparent way in comparison to the treatments developed originally (see [59]).

2. Self-attracting random walks

2.1 Introduction

We discuss in this chapter a number of problems of random walks with self-attracting path interactions which are all closely related to large deviation theory. A simple case of an attraction would be just to change sign in the (weakly) self repellent case of Chapter 1. For technical reasons, it is convenient to work with continuous time but discrete state space Markov processes. Therefore, we consider the standard symmetric random walk on $\mathbb{Z}^d$ starting in 0 having holding times with expectation $1/d$. The path measure on the space $D_\infty = D([0,\infty), \mathbb{Z}^d)$ of right continuous piecewise constant paths is denoted by P. We will also write D_T for the set of paths of length T. As usual, we write $X_t(\omega) = \omega_t$, $\omega \in D_\infty$ for the evaluation map. We then transform the path measure in the same way as in the weakly repellent case, just having the opposite sign of the coupling constant:

$$\hat{P}_{T,\beta}(d\omega) \stackrel{\mathrm{def}}{=} \exp\left[\beta \int_0^T ds \int_0^T dt 1_{\omega_t=\omega_s}\right] P(d\omega) \Big/ Z_{T,\beta}, \quad \beta > 0,$$

However, it is easy to see that this is not an interesting object, as the self-attraction is far too strong. In fact, a path staying just all the time at 0 up to time T gets a weight $\exp\left[\beta T^2\right]$, whereas the entropic cost for doing that is only of order $\exp\left[-cT\right]$. It is therefore evident that as $T \to \infty$ the path measures just concentrates with probability going to 1 on the path identical to 0. A more interesting object is obtained when having the interaction only of order $1/T$. Therefore, we define for $\beta > 0$:

$$\hat{P}_{T,\beta}(d\omega) \stackrel{\mathrm{def}}{=} \exp\left[\frac{\beta}{T} \int_0^T ds \int_0^T dt\, 1_{\omega_t=\omega_s}\right] P(d\omega) \Big/ Z_{T,\beta}. \tag{2.1}$$

This path measure has been investigated in two papers [24] and [14]. In the first one, it was shown that the for $d \geq 2$, the measure behaves diffusively if β is small enough (actually for discrete time walks), and in the second, it was shown that for $d = 1$, and in all dimensions if β is large enough, the path measures is localized in the sense that the end points ω_T have fluctuations of order one, but these fluctuations stay non-trivial in the $T \to \infty$ limit.

Therefore, for $d \geq 2$, there is what is called a collapse transition if β grows from small values to large ones. We will give the argument for the diffusive behavior in Section 2, and discuss the localized phase in Section 3.

There are other models which have a similar behavior. One case is Brownian motion transformed by the Wiener sausage in such a way that large volumes of the sausage are suppressed. For a random walk the rôle of volume of the sausage is played by the number $N_T(\omega)$ of sites visited up to time T, and for these, this would correspond in transforming the path measure P in the following way:

$$d\hat{P}_{T,\beta}(\omega) \stackrel{\mathrm{def}}{=} \exp\left[-\beta N_T(\omega)\right] dP(\omega)/\, Z_{T,\beta}, \tag{2.2}$$

where

$$Z_{T,\beta} \stackrel{\mathrm{def}}{=} E\left(\exp\left[-\beta N_T(\omega)\right]\right).$$

It had been proved in [11] (and in [72] for the Wiener sausage) that for $d = 2$ the path measure is concentrated on paths which stay inside a ball of radius of order $T^{1/4}$. This is closely related to the classical analysis of Donsker and Varadhan of the leading order asymptotic behavior of $Z_{T,\beta}$. Sznitmans results and techniques have been extended recently to arbitrary dimension by Povel [62]. We will give a discussion of these results in Section 2.5. Sznitman's approach uses the enlargement of obstacles techniques (see [73]). The approach in [11] is more combinatorial by "path counting", and is rather involved. The problem amounts to a droplet construction, where the macroscopic droplet is trivial, namely just a ball. It is remarkable that one can prove that in all dimensions the microscopic droplet approaches the macroscopic one in $L_\infty-$norm (at least from "outside"), whereas the corresponding analytic variational problem is stable only in L_1 (for $d \geq 3$). We will not be able to present the details here, but we will give a discussion of this aspect.

This model has no collapse transition: For all $\beta > 0$, the path measures lives on a droplet of scale $T^{1/(d+2)}$. However, an interesting and somewhat unexpected features shows up if we make the self-attraction weaker by replacing β by a coupling constant which goes to 0 as $T \to \infty$. Fix $\alpha > 0$ and define

$$Z_{T,\beta,\alpha} \stackrel{\mathrm{def}}{=} E\left(\exp\left[-\frac{\beta}{T^\alpha} N_T(\omega)\right]\right).$$

One way to estimate this is just by Jensen's inequality, which gives the trivial estimate

$$Z_{T,\beta,\alpha} \geq \exp\left[-\frac{\beta}{T^\alpha} E\left(N_T(\omega)\right)\right].$$

It is well know that for $d \geq 3$, asymptotically $E\left(N_T(\omega)\right) \sim d\kappa T$, where κ is the escape probability for a discrete time random walk from a single point.

(The factor d is coming from the holding times having expectation $1/d$). For $\alpha = 0$, this estimate is very bad as it is known from the classical work of Donsker and Varadhan [34] that $Z_T \approx \exp\left[-cT^{d/(d+2)}\right]$. It turns out that the Jensen estimate is essentially sharp as soon as $\alpha > 2/d$. Similar to (2.2), we can define a path measure

$$\widehat{P}_{T,\beta,\alpha}\left(d\omega\right) \stackrel{\mathrm{def}}{=} \exp\left[-\beta T^{-\alpha} N_T(\omega)\right] P\left(d\omega\right) / \, Z,$$

but these measures have not (yet) been investigated in the literature. The fact that the Jensen estimate is essentially sharp for $\alpha > 2/d$ suggests that this path measure is just diffusive in this regime. As α crosses $2/d$, there is a collapse transition: Jensen's inequality is no longer sharp and in fact

$$E\left(\exp\left[-\beta T^{-\alpha} N_T(\omega)\right]\right) \stackrel{\log}{\sim} \exp\left[-\mathrm{const} \times T^{(d-2\alpha)/(2+d)}\right], \tag{2.3}$$

$\stackrel{\log}{\sim}$ meaning that the quotient of the logarithms tends to 1. This is been proved in [10] and [70]. The somewhat strange exponent of T will become clear in Section 2.5. The Povel result suggests, but this has not been proved, that for $\alpha < 2/d$, the path measure is localized on scale $T^{\frac{1+\alpha}{2+d}}$, and for $\alpha > 2/d$, it is just diffusive. Remark that the critical case $\alpha = 2/d$ $(d \geq 3)$, would correspond to the path measure living on a subdiffusive scale $T^{1/d}$. This critical case has recently been investigated in [6] and there are some quite interesting features. For instance, it turns out that there is a collapse transition from small to large β. I will discuss this critical case in Section 2.6, but again, up to now, the path measures have not been investigated, but only the "free energy", i.e. an asymptotic evaluation of the type (2.3).

There are several motivations for the investigation of these problems. In the physical literature, the main interest in collapse transitions are for models which have a mixed attractive and repulsive interaction. Mathematically, essentially nothing is known, not even about the diffusive behavior in high dimensions. For the physical background, see [20], [21]. As an example, consider the interactive random walk (in discrete time, say), defined by

$$\hat{P}_{n,\beta,\gamma}(\omega) = \frac{1}{Z_{n,\beta,\gamma}} \exp\left[-\beta \sum_{1 \leq i < j \leq n} 1_{\omega_i = \omega_j} + \gamma \sum_{1 \leq i < j \leq n} 1_{|\omega_i - \omega_j| = 1}\right],$$

$\beta, \gamma > 0$. One would expect that if $\gamma \ll \beta$, and at least in high dimension, the repulsion dominates the attraction, and the model would just be diffusive. There is however no proof of this, and it appears that the lace expansion with which the diffusive behavior for $\gamma = 0$ has been proved is completely powerless as soon as there is a positive γ. In the physical literature, there is a collapse transition predicted if one changes the parameter γ.

Some of the investigations above had been motivated by the long-standing open problem to determine the effective mass of the so-called polaron in the strong coupling regime, which is a (one-dimensional) model with a Kac-type potential and a continuous symmetry (actually a shift degeneracy). This is closely related with some of the models discussed here, but it is still (mathematically) an open problem. For a heuristic derivation, see [68]. We will give a short discussion of this in the final section of this chapter.

It is not possible to give proofs of all the results presented and I will concentrate on those parts which I think are the most instructive ones. In particular, I give a proof of the probabilistic part of the evaluation of the limit of the model (2.1). The proof presented here incorporates some simplifications of the one given in the original paper [14].

To finish this introduction, I state some of the relevant facts on large deviations, which will be used during this and the next chapter. For proofs, see e.g. [43].

Let Y_T, $T \geq 0$, be a stochastic process, where T is either $\in \mathbb{N}$ or $\in \mathbb{R}$, which takes values in a Polish space Σ. Let further $(a_T)_{T\geq 0}$ be an increasing, real-valued, positive function of T. One says that (Y_T) satisfies an (a_T)-**large deviation principle (LDP)** with rate function $I : \Sigma \to [0, \infty]$, if the following properties are satisfied:

L1 I is upper semi-continuous and has the property that for any $a > 0$, the set $\{x \in \Sigma : I(x) \leq a\}$ is compact in Σ.

L2 For any closed subset $A \subset \Sigma$, one has
$$\limsup_{T\to\infty} a_T^{-1} \log P(Y_T \in A) \leq - \inf_{x\in A} I(x).$$

L3 For any open subset $A \subset \Sigma$, one has
$$\liminf_{T\to\infty} a_T^{-1} \log P(Y_T \in A) \geq - \inf_{x\in A} I(x).$$

There are a number of cases, where only a weaker form is valid, namely where L1 is replaced by just the semi-continuity condition, and L2 is required only for compact subsets A of Σ. In such a case one says that a **weak large deviation principle** holds.

A consequence of a LDP is the following result, called Varadhan's Lemma

Lemma 2.14 (Varadhan). *Assume that (Y_T) satisfies an (a_T)-LDP with rate function I. Then for any continuous function $F : \Sigma \to \mathbb{R}$ which is bounded above, one has*
$$\lim_{T\to\infty} \frac{1}{a_N} \log E(\exp(a_T F(Y_T))) = \sup_{x\in\Sigma} (F(x) - I(x)).$$

Remark 2.15. If F is only upper semi-continuous, then one gets an upper bound for the limit superior, and if F is lower-semicontinuous, one gets a lover bound for the limit inferior.

Here the examples which are important for us. The first is the Sanov-Theorem:

Let X_t, $t \in \mathbb{N}$, be a sequence of i.i.d. random variables, taking values in a Polish space S, with law μ. The so-called empirical process is defined by

$$L_T \stackrel{\text{def}}{=} \frac{1}{T} \sum_{t=1}^{T} \delta_{X_t} .$$

L_T takes values in the space $\mathcal{M}_1^+ (S)$ with the weak topology. $\mathcal{M}_1^+ (S)$ is a Polish space itself (more precisely, there is a metric, e.g. the Prohorov metric, which metricizes the weak topology, and with which the space is Polish). Then we have

Theorem 2.16 (Sanov). *(L_T) satisfies a T-LDP with rate function I given by*

$$I(\nu) = k(\nu \mid \mu) \stackrel{\text{def}}{=} \int \log\left(\frac{d\nu}{d\mu}\right) d\nu,$$

where $I(\nu)$ is defined to be ∞ if ν is not absolutely continuous with respect to μ or if the logarithm of the derivative is not integrable.

Another case of great importance for us is the celebrated large deviation theorem for the Brownian motion by Donsker and Varadhan. For this, we consider a d-dimensional Brownian $(\beta_t)_{t\geq 0}$, and again the empirical distribution

$$L_T \stackrel{\text{def}}{=} \frac{1}{T} \int_0^T \delta_{\beta_t}\, dt.$$

Theorem 2.17 (Donsker-Varadhan). *(L_T) satisfies a weak T-LDP with rate function I given by*

$$I(v) \stackrel{\text{def}}{=} \frac{\|\nabla f\|_2^2}{2},$$

where $f = \sqrt{d\nu/dx}$. $I(\nu)$ is defined to be ∞ if ν is not absolutely continuous with respect to Lebesgue measure, or when f is not (weakly) differentiable with gradient in L_2.

The fact that the empirical distribution satisfies only a weak LDP causes a lot of problems. One way out is often by a compactification procedure. The fact is that the Brownian motion on a compact manifold satisfies a full LDP. The case which is important for us will be the Brownian motion on a flat torus, i.e. just the Brownian motion which is wound up periodically.

Convention: During this chapter, we again use C as a generic positive constants not necessarily the same at different occurrences. It may depend on the dimension, and on a fixed coupling constant β, but on nothing else, except when indicated clearly. In contrast, we use $c_1, c_2, \ldots$ for positive constants which stay fixed after having been introduced.

2.2 A maximum entropy principle

To start with, we consider the following trivial problem. Let $X_1, X_2, \ldots$ be a sequence of independent coin tossings: $P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$. If $\alpha > \frac{1}{2}$, then by the Bernoulli law of large numbers

$$P(S_n/n \geq \alpha) \to 0,$$

as $n \to \infty$, where $S_n = \sum_{i=1}^n X_i$. Question: what is the limiting distribution of X_1, if we condition on the event $\{S_n/n \geq \alpha\}$? The answer is evident:

$$\lim_{n\to\infty} P(X_1 = 1 \mid S_n/n \geq \alpha) = \alpha.$$

Similarly, the conditional distribution of $X_1, \ldots, X_{k(n)}$ converges (in total variation) to coin tossing if $k(n) = o(n)$. (This can of course not be true for $k(n) = n$).

We consider a slightly more general problem. We assume that the X_i are i.i.d. random variables, taking values in a Polish space S equipped with its Borel field $\mathcal{S}$. P is the product measure of the law μ_o of the X_i on $\Omega = (S, \mathcal{S})^{\mathbb{N}}$, with the X_i being the projections $\Omega \to S$. The empirical distribution is

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}.$$

Let further $F : \mathcal{M}_1^+(S) \to [-\infty, \infty)$ be an upper semicontinuous function. We will assume that F is bounded above, but it may take the value $-\infty$. We consider the transformed measure on Ω

$$d\hat{P}_n = \frac{1}{Z_n} \exp\left[nF(L_n)\right] dP.$$

$$Z_n = E\left(\exp[nF(L_n)]\right)$$

By Sanov's Theorem, and the upper semicontinuity of F, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n \leq b_F \stackrel{\text{def}}{=} \sup_\mu \left[F(\mu) - \int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu \right], \tag{2.4}$$

and if F_{lc} is the lower semi-continuous modification of F, then

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_n \geq b_{F_{\mathrm{lc}}}. \tag{2.5}$$

Proposition 2.18. *Assume $b_F = b_{F_{\mathrm{lc}}} > -\infty$ (which in particular is true if F is continuous). Then the sequence $\{\hat{P}_n\}$ is tight in the weak topology on the set of probability measures on Ω. Any limiting probability measure Q has a representation $Q = \int \mu^{\mathbb{N}} \Gamma(d\mu)$, where Γ is a probability measure on $\mathcal{M}_1^+(S)$ which is concentrated on*

$$K_F = \left\{ \mu : F(\mu) - \int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu = b_F \right\}$$

Remark 2.19. This is a very weak formulation of a so called propagation of chaos result. For much stronger variants (under more restrictive conditions on F), see [5].

Proof. The proof is a very easy application of the Sanov Theorem. As the rate function has compact level sets, it follows that K_F is a compact subset of $\mathcal{M}_1^+(S)$. Moreover, if $U_\varepsilon(K_F)$ is an open neighborhood of K_F, it follows from (2.4), (2.5) and the assumption $b_F = b_{F_{\mathrm{lc}}}$ that

$$\lim_{n\to\infty} \hat{P}_n(L_n \in U_\varepsilon(K_F)) = 1. \tag{2.6}$$

In fact, $\hat{P}_n(L_n \notin U_\varepsilon(K_F)) = E\left(\exp\left[nG(L_n)\right]\right)/\, E\left(\exp\left[nF(L_n)\right]\right)$, where we set $G \stackrel{\text{def}}{=} F$ on $\left(U_\varepsilon(K_F)\right)^c$ and $-\infty$ otherwise. Then the denominator behaves in leading order as $\exp\left[nb_F\right]$, whereas the numerator can be estimated from above in leading order by $\exp\left[n \sup_{\mu\notin U_\varepsilon(K_F)} \left[F(\mu) - \int \log\left(d\mu/d\mu_o\right) d\mu\right]\right] \ll \exp\left[nb_F\right]$.

From (2.6) it follows that the sequence $\left(\hat{P}_n L_n^{-1}\right)_{n\geq 1}$ of probability measures on $\mathcal{M}_1^+(S)$ is tight and any limit measure is supported by K_F. Now, we decompose

$$\hat{P}_n\left(\cdot\right) = \int_{\mathcal{M}_1^+(S)} \hat{P}_n\left(\cdot|L_n\right) d\left(\hat{P}_n L_n^{-1}\right).$$

Evidently, we have $\hat{P}_n\left(\cdot|L_n\right) = P_n\left(\cdot|L_n\right)$, which is just drawing without replacement. It is well know that for large n, drawing without replacement is asymptotically the same as drawing with replacement, if we consider only $o(n)$ drawings (which is much more than we need for weak topology considerations). Therefore, in the weak topology (and also in some stronger ones), $P_n\left(\cdot|L_n\right)$ is close to $L_n^{\mathbb{N}}$. From this, we easily see that the sequence $\{\hat{P}_n\}_{n\geq 1}$ is tight (as a sequence of probability measures on Ω), and every limit point is of the required form.

The above Proposition evidently applies to the coin tossing example at the beginning. The empirical distribution there is just the relative number of 1's in the sequence, and we take $F = 0$ if this is $\geq \alpha$, and $-\infty$ otherwise. It should however be remarked that already quite simple modification of this trivial example can become quite delicate, as is revealed by the following example (see [29]):

Exercise 2.20. Start with the coin tossing sequence of length n as above, and define

$$T_n = \sum_{i=1}^{n-1} 1_{\{X_i=1, X_{i+1}=1\}}.$$

Then determine

$$\lim_{n\to\infty} P(X_1 = 1 \,|\, T_n/n \geq \alpha)$$

for $\alpha > 1/4$.

The exercise falls into a category of problems running under the heading "equivalence of ensembles", in that case between some type of microcanonical and grand canonical ones. There are still many open problems in this field (see for instance [53]).

Remark 2.21. If K_F contains just one point, say μ, then the Proposition states that $\hat{P}_n$ converges to $\mu^{\mathbb{N}}$. If K_F contains more than one point, then one usually has to derive finer asymptotics in order to evaluate the limit law of $\hat{P}_n$. The situation we encounter in some of the following sections is more delicate than the one in Proposition 2.18, mainly because there K_F contains more than one point (and is not even compact).

The models we discuss in this (and to some extent also in the next) chapter are all variations of the above situation: The path measure of a "simple" process is transformed by a density of the form

$$\exp\left[H\left(\text{path}\right)\right] dP/E\exp\left[H\left(\text{path}\right)\right],$$

where the "Hamiltonian" H is given by some self-interaction of the path (or in the next chapter by some interaction with a "wall". What large deviation theory in these examples provides, is an asymptotic evaluation of the normalizing $E\exp\left[H\left(\text{path}\right)\right]$, usually up to logarithmic equivalence. This is usually far from sufficient to determine exactly the path measure, which will be the main task in some of the next sections. The above Proposition 2.18 is in this respect a bit misleading.

Let us now start with discussing the self-attracting random walk. P is the law of the standard symmetric random walk (in continuous time), starting at 0, with holding times of expectation $1/d$, and we define the transformed path measure $\hat{P}_{T,\beta}$ by (2.1). It is formally convenient to have $\hat{P}_{T,\beta}$ defined as a measure on paths of infinite length, i.e. on $D_\infty = D\left([0,\infty), \mathbb{Z}^d\right)$. Of course, after time T it is just an ordinary random walk. Remark that the Hamiltonian $\frac{1}{T}\int_0^T ds \int_0^T dt\, 1_{\omega_t=\omega_s}$ can be written as $T\left\|l_T\right\|_2^2$ where l_T is the normalized local time:

$$l_T(x) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T 1_{\{X_s=x\}} ds,$$

and $\left\|l_T\right\|_2^2 = \sum_x l_T(x)^2$. Clearly, l_T is a random probability measure on $\mathbb{Z}^d$. It satisfies a weak LDP (see e.g. [30]):

Proposition 2.22. $(l_T)_{T\geq 0}$ *satisfies a weak LDP in* $\mathcal{M}_1^+\left(\mathbb{Z}^d\right)$ *with rate function* $I(\mu)=\frac{1}{2}\sum_{\langle x,y\rangle}(\sqrt{\mu(x)}-\sqrt{\mu(y)})^2$, *where summation is over (unordered) nearest neighbor pairs* x,y.

From this proposition, we easily get:

Proposition 2.23.

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log Z_{T,\beta}&=\lim_{T\to\infty}\frac{1}{T}\log E\exp\left[\beta T\left\|l_T\right\|_2^2\right] \\ &=b(\beta)\overset{\text{def}}{=}\sup_{\mu}\left[\beta\sum_x\mu(x)^2-I(\mu)\right].\end{aligned}\tag{2.7}$$

Proof. This is essentially Varadhan's Lemma but there is a slight problem. If $F:\mathcal{M}_1^+(\mathbb{Z}^d)\to\mathbb{R}$ is continuous, and has the property that $\{\mu:F(\mu)\geq a\}$ is compact for all a, then by a version of Varadhan's Lemma we get

$$\lim_{T\to\infty}\frac{1}{T}\log E\exp\left[TF(l_T)\right]=\sup_{\mu}\left(F(\mu)-I(\mu)\right).$$

In our case, we take $F(\mu)=\sum_x\mu(x)^2$, but this evidently does not satisfy the above compactness property. There is however a very simple trick. Consider the periodized situation, where we replace $\mathbb{Z}^d$ by a finite discrete torus $T_R=\{0,\dots,R-1\}^d$, and correspondingly a symmetric random walk with periodic boundary conditions on this torus. We can just map the old random walk by "winding it up" in an evident way. Then we have

$$\left\|l_T\right\|_2^2\leq\left\|l_T^R\right\|_2^2,\tag{2.8}$$

where $l_T^R(x)$ is the local time for the wound up random walk on the torus. Now, for the random walk on the torus, we evidently have a full LDP, because $\mathcal{M}_1^+(T_R)$ itself is compact, with a rate function $I^R(\mu)=\frac{1}{2}\sum_{\langle x,y\rangle}(\sqrt{\mu(x)}-\sqrt{\mu(y)})^2$, the only difference being that the summation is now over nearest neighbors on the torus. Therefore

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\log Z_{T,\beta}&\leq\limsup_{T\to\infty}\frac{1}{T}\log E\exp\left[\beta T\left\|l_T^R\right\|_2^2\right] \\ &=b^R(\beta)\overset{\text{def}}{=}\sup_{\mu}\left[\beta\sum_x\mu(x)^2-I^R(\mu)\right],\end{aligned}$$

and it is easy to see that $\lim_{R\to\infty}b^R(\beta)=b(\beta)$. Therefore, we get

$$\limsup_{T\to\infty}\frac{1}{T}\log Z_{T,\beta}\leq b(\beta).$$

The lower bound is no problem and follows from the weak LDP (and the continuity of the functional).

It should be remarked that the above monotonicity argument is rather special. It depends crucially on (2.8). We will encounter in Section 2.4 a situation where such a procedure cannot immediately be applied, and where things become then more delicate. Having the above large deviation property, a natural first question is to ask whether or not there are minimizers of the variational problem. This is directly connected with the question if $b(\beta) > 0$.

Proposition 2.24. a) *If* $d = 1$, *then* $b(\beta) > 0$ *for all* $\beta > 0$.
b) *If* $d \geq 2$ *then there exists* $\beta_{cr}(d) > 0$ *such that* $b(\beta) > 0$ *for* $\beta > \beta_{cr}(d)$ *and* $b(\beta) = 0$ *for* $\beta < \beta_{cr}(d)$.

Proof. Evidently, $b(\beta)$ is increasing in β, and furthermore, $b(\beta) > 0$ if β is large enough. This simply follows from the fact that $I(\delta_0)$ is finite. Therefore, it remains to prove that for $d = 1$, we have $b(\beta) > 0$ for all β, and that for $d \geq 2$

$$\sum_x \mu(x)^2 \leq CI(\mu). \tag{2.9}$$

We start with the one dimensional case. We define a sequence of measures which become flat and flatter:

$$\mu_n(x) = \frac{\max(1 - |x|/n, 0)^2}{\xi_n},$$

where ξ_n is the appropriate norming. Evidently, $\xi_n \sim Cn$. Therefore, $\sum_x \mu_n(x)^2 \sim C/n$, and $I(\mu_n) \sim Cn^{-2}$. Therefore, $\beta \sum_x \mu_n(x)^2 > I(\mu_n)$ for any $\beta > 0$ if n is large enough. This proves a).

The inequality (2.9) follows from the (discrete version of the) Sobolev inequality

$$\|g\|_4^4 \leq C \|g\|_2^2 \|\nabla g\|_2^2,$$

applied to $\mu(x) = g^2(x)$, which holds for $d \geq 2$. Here ∇ denotes the discrete gradient.

It turns out that if $b(\beta) > 0$, then there exist solutions of the variational problem. Let

$$K_\beta \stackrel{\text{def}}{=} \left\{ \mu \in \mathcal{M}_1^+(\mathbb{Z}^d) : \beta \sum_x \mu(x)^2 - I(\mu) = b(\beta) \right\}. \tag{2.10}$$

One of the basic difficulties we will encounter is that K_β is shift invariant: Any shift of an element of K_β is again in K_β. We summarize the basic facts about this set.

Proposition 2.25. *Assume $b(\beta) > 0$. Then*

a) $K_\beta \neq \emptyset$.
b) *Any $\mu \in K_\beta$ satisfies $\mu(y) > 0$ for all $y \in \mathbb{Z}^d$.*
c) *There exist $C > 0$ such that for any $\mu \in K_\beta$ there exists $x_\mu \in \mathbb{Z}^d$ with $\mu(y - x_\mu) \leq C \exp[-|y|/C]$ for all y.*

The proof is not difficult, but a bit lengthy. I will not give it here (see [14]).

A natural question is if there is uniqueness modulo shifts as soon as $b(\beta) > 0$. Unfortunately, I don't know the answer, not even for $d = 1$. Corresponding uniqueness questions for variational problems in the continuous setting on $\mathbb{R}^d$ have a long history with many results. However, the knowledge about similar questions on $\mathbb{Z}^d$ is essentially zero. One of the difficulties in the discrete situation is that standard symmetrization techniques do not work. The discrete problems seem to be inherently more delicate than the continuous ones. Take for instance the variational problem in the one-dimensional case, but in the continuous situation. This just is the problem to maximize

$$\beta \int g(x)^4 dx - \frac{1}{2} \int g'(x)^2 dx,$$

subject to the condition $\int g(x)^2 dx = 1$. It is easy to see that modulo shifts, there is just one solution of the Euler equation

$$4\beta g(x)^3 + g''(x) = \lambda g(x) \tag{2.11}$$

which decays to 0 at infinity and satisfies $\int g(x)^2 dx = 1$ (just $\sqrt{\beta/2}/\cosh(\beta x)$ and its shifts). On the other hand, the Euler equation of the discrete problem, namely

$$4\beta g(x)^3 + \Delta g(x) = \lambda g(x) \tag{2.12}$$

subject to $\sum g(x)^2 = 1$ (we have replace $\mu(x)$ by $g(x)^2$) has countably many such solutions. Δ is the discrete Laplacian. I have no formal proof of this, but playing on the computer one "sees" them, and it is probably not difficult to prove it. (Computer simulations however indicate that among these solution there are just two candidates as maximizers. Both are symmetric, although we don't have a proof that the maximizers have to be symmetric.). Anyway, one easily gets convinced that (2.12) is a much more delicate equation than (2.11), and even more so the discrete variational problem.

It is not difficult to see that one has uniqueness if β is large enough. This is just coming from the fact that for "$\beta = \infty$", the solutions are unique modulo shifts, namely just the δ_x. By a perturbation argument around $\beta = \infty$ one can prove that uniqueness persists for large β:

Proposition 2.26. *If* $\beta \geq 2d$ *then*

$$K_\beta = \left\{\theta_x \mu : x \in \mathbb{Z}^d\right\}$$

for some $\mu \in \mathcal{M}_1^+(\mathbb{Z}^d)$, *where* $\theta_x : \mathcal{M}_1^+(\mathbb{Z}^d) \to \mathcal{M}_1^+(\mathbb{Z}^d)$ *is the usual shift* $\theta_x \mu(y) = \mu(y - x)$.

This is Proposition 1.19 of [14]. I will not repeat the proof here. There is actually nothing special about $2d$. The proof allows for slightly lower values of β. (The proof in [14] can actually be simplified somewhat, if one is satisfied with a statement "for large enough β ...".)

In view of the Proposition 2.18, it is natural to conjecture that for $\beta > \beta_{cr}(d)$, one has that $\hat{P}_{T,\beta}$ behaves for large T such that l_T is close to some element in K_β. This is in fact true (see Proposition 2.35 below). What makes things delicate is that K_β contains infinitely many elements. It will turn out that there are infinitely many elements of K_β which will get positive limiting weight under $\hat{P}_T l_T^{-1}$. In the case where one has uniqueness modulo shifts, we will actually prove that all elements of K_β get positive weight. However, for $\mu \in K_\beta$ which lie for out, these weight will be small, uniformly in T. A preformulation of the main result on this collapsed phase is the following:

Theorem 2.27. *Assume* $b(\beta) > 0$. *Then*

a) $\left(\hat{P}_{T,\beta}\, l_T^{-1}\right)_{T>0}$ *is tight in* $\mathcal{M}_1^+(\mathcal{M}_1^+(\mathbb{Z}^d))$.

b) *There exists* $c(\beta) > 0$ *such that*

$$\sup_T \int \exp\left[c(\beta)\, \|\omega_T\|\right] d\hat{P}_{T,\beta} < \infty.$$

c) *If there is uniqueness modulo shift, then*

$$\lim_{T\to\infty} \hat{P}_{T,\beta} l_T^{-1},\ \lim_{T\to\infty} \hat{P}_{T,\beta}\, \omega_T^{-1} \quad \textit{and} \ \lim_{T\to\infty} \hat{P}_{T,\beta}$$

exist.

The exact formulation of the limits needs a bit of preparation, and we will give it later. $\lim_{T\to\infty} \hat{P}_{T,\beta}$ is understood in the sense of weak convergence on the path space. In particular, it does not imply the existence of $\lim_{T\to\infty} \hat{P}_{T,\beta}\, \omega_T^{-1}$ which has to be treated separately. The latter is the more interesting object. The existence of the limit means that there is no rescaling for ω_T, so that this random variable stays stochastically bounded under $\hat{P}_{T,\beta}$ as $T \to \infty$.

It is natural to conjecture that for $\beta < \beta_{cr}$, $\hat{P}_{T,\beta}$ just behaves diffusively, but there is no full proof of that. What Brydges and Slade in [24] proved is that for $d \geq 2$ there exists $\beta_o(d) \leq \beta_{cr}(d)$ such that for $\beta < \beta_o(d)$ there is diffusive behavior (with some complications for $d = 2$). It could actually well be that the $\beta_o(d)$ they define is exactly $\beta_{cr}(d)$, but I don't know how to prove this. We will threat the diffusive behavior in the next section. In Section 2.4, we will then come to the large β i.e. collapsed case.

2.3 The diffusive phase for self-attracting random walks

I am presenting part of the arguments in [24] for the existence of a diffusive phase for dimensions $d \geq 2$. The two dimensional case is the most interesting one, and it is related to the topics discussed in Chapter 1. I give a detailed discussion of the case $d \geq 3$, and will add some comments about the two-dimensional case. For abbreviation, we set

$$\gamma_T = \frac{1}{T} \int_0^T dt \int_0^T ds 1_{\omega_s = \omega_t} = T \left\| l_T \right\|_2^2.$$

Let $p_s(y)$ be the transition probabilities for our random walk. Then one has the estimates

$$p_s(y) \leq C \min \left[1, |s|^{-d/2} \right] \exp \left[-|y|/Cs \right] \tag{2.13}$$

and

$$G(y) \stackrel{\text{def}}{=} \int_0^\infty p_s(y) ds \leq C \min \left(|y|^{-d+2}, 1 \right) \tag{2.14}$$

for $d \geq 3$. ((2.13) is actually very crude, but it suffices for our purpose).

Lemma 2.28. *Assume* $d \geq 3$.

a) *There exists* $\beta_o(d) > 0$ *such that*

$$\sup_{T>0} E \left(\exp \left[\beta \gamma_T \right] \right) < \infty$$

for $\beta < \beta_o$.

b) $E \left(\gamma_T - E\gamma_T \right)^2 = o(1)$ *as* $T \to \infty$.

Proof. a) By Jensen's inequality, we have

$$\begin{aligned}
&\exp \left[\beta \gamma_T \right] \\
&\leq \sum_y l_T(y) \exp \left[\beta T l_T(y) \right] = \frac{1}{T} \sum_y \sum_{n=0}^\infty \frac{\beta^n}{n!} \left(\int_0^T ds 1_{\omega_s = y} \right)^{n+1} \\
&= \frac{1}{T} \sum_y \sum_{n=0}^\infty \beta^n (n+1) \int_{0 \leq s_1 < \ldots < s_{n+1} \leq T} 1_{\left\{ \omega_{s_1} = y, \ldots, \omega_{s_{n+1}} = y \right\}} ds_1 \ldots ds_{n+1}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&E\left(\exp\left[\beta\gamma_T\right]\right)\\
&\leq \frac{1}{T}\sum_y\sum_{n=0}^{\infty}\beta^n(n+1)\int_{0\leq s_1<\ldots<s_{n+1}\leq T} p_{s_1}(y)p_{s_2-s_1}(0)\ldots p_{s_{n+1}-s_n}(0)\\
&=\frac{1}{T}\sum_{n=0}^{\infty}\beta^n(n+1)\int_{0\leq s_1<\ldots<s_{n+1}\leq T} p_{s_2-s_1}(0)\ldots p_{s_{n+1}-s_n}(0)\\
&\leq\sum_{n=0}^{\infty}\beta^n(n+1)G(0)^n,
\end{aligned}
$$

which is finite if $\beta < G(0)^{-1}$. This proves a)

In order to prove b), remark first that

$$
\begin{aligned}
E\gamma_T &= \frac{2}{T}\sum_y\int_0^T ds\int_s^T dt\, p_s(y)p_{t-s}(0)\\
&=\frac{2}{T}\int_0^T ds\int_s^T dt\, p_{t-s}(0) = 2G(0)+o(1).
\end{aligned}
$$

Therefore, we have to prove

$$
\begin{aligned}
&E\left(\gamma_T^2\right) = 4G(0)^2 + o(1).\\
&\quad\gamma_T^2 = \frac{4}{T^2}\int_{0\leq s_1<t_1\leq T} ds_1\, dt_1\int_{0\leq s_2<t_2\leq T} ds_2\, dt_2\, 1_{\omega_{s_1}=\omega_{t_1}}1_{\omega_{s_2}=\omega_{t_2}}.
\end{aligned}
$$

When calculating the expectation, we have to distinguish between the cases where the two intervals $[s_1,t_1]$ and $[s_2,t_2]$, are disjoint, one contains the other, and when they nontrivially overlap, respectively. The first one is the main contribution:

$$
\frac{8}{T^2}\int_{0\leq s_1<t_1\leq s_2<t_2\leq T} ds_1\, dt_1\, ds_2\, dt_2\, p_{t_1-s_2}(0)p_{t_2-s_2}(0) = 4G(0)^2 + o(1).
$$

It is readily checked that the other contributions are negligible:

$$
\begin{aligned}
R_1(T) &\stackrel{\text{def}}{=} \frac{1}{T^2}\int_{0\leq s_1\leq s_2<t_2\leq t_1\leq T} ds_1\, dt_1\, ds_2\, dt_2 E\left(1_{\omega_{s_1}=\omega_{t_1}}1_{\omega_{s_2}=\omega_{t_2}}\right) = o(1)\\
R_2(T) &\stackrel{\text{def}}{=} \frac{1}{T^2}\int_{0\leq s_1\leq s_2<t_1\leq t_2\leq T} ds_1\, dt_1\, ds_2\, dt_2 E\left(1_{\omega_{s_1}=\omega_{t_1}}1_{\omega_{s_2}=\omega_{t_2}}\right) = o(1).
\end{aligned}
$$

We check this for the last case.

$$
R_2(T) = \frac{1}{T^2}\sum_y\int_{0\leq s_1\leq s_2<t_1\leq t_2\leq T} ds_1\, dt_1\, ds_2\, dt_2\, p_{s_2-s_1}(y)p_{t_1-s_2}(y)p_{t_2-t_1}(y).
$$

For $d \geq 4$, we can estimate the r.h.s. by $(1/T) \sum_y G(y)^3 = O(1/T)$, but for $d = 3$, this is divergent, and one has to argue slightly more carefully. Using (2.13), one gets

$$\begin{aligned} R_2(T) &\leq \frac{C}{T} \sum_y \left(\int_0^T p_s(y) ds \right)^3 \leq \frac{C}{T} \sum_{y \neq 0} \min \left(\frac{1}{|y|^3}, \exp \left[-\frac{|y|}{CT} \right] \right) \\ &= O\left(\frac{\log T}{T} \right). \end{aligned}$$

This proves the Lemma.

With this Lemma, one now easily gets the following result:

Theorem 2.29. *Assume $d \geq 3$ and $\beta < \beta_o(d)$. Then, using Brownian scaling,*

$$\rho_T(\omega)(t) \stackrel{\text{def}}{=} \omega(tT)/\sqrt{T},\ \omega \in D\left([0,\infty), \mathbb{R}^d\right),$$

one has

$$\lim_{T \to \infty} \hat{P}_{T,\beta} \rho_T^{-1} = P_\infty,$$

weakly, where P_∞ is the standard Wiener measure.

Proof. This is immediate from the estimates in the Lemma 2.28: Let Φ : $D\left([0,\infty), \mathbb{R}^d\right) \to \mathbb{R}$ be continuous and bounded. Then

$$\begin{aligned} \lim_{T \to \infty} \int \Phi \, d\left(\hat{P}_{T,\beta} \rho_T^{-1} \right) &= \lim_{T \to \infty} \frac{\int \exp\left[\beta \gamma_T\right] \left(\Phi \circ \rho_T \right) dP}{\int \exp\left[\beta \gamma_T\right] dP} \\ &= \lim_{T \to \infty} \frac{\int \exp\left[\beta(\gamma_T - E(\gamma_T))\right] \left(\Phi \circ \rho_T \right) dP}{\int \exp\left[\beta(\gamma_T - E(\gamma_T))\right] dP} \\ &= \lim_{T \to \infty} \int \left(\Phi \circ \rho_T \right) dP = \int \Phi dP_\infty. \end{aligned}$$

The third equality is coming from the fact that $\gamma_T - E_T(\gamma_T)$ converges to 0 in probability according the Lemma 2.28 b), and the necessary exponential moment estimate are from Lemma a).

Remark 2.30. It is also not difficult to prove that one has convergence of all moments of finite dimensional distributions (see [24]).

The two dimensional case is more delicate and the limiting measure is more interesting. I will give only some short comments about this case. We step back to the discussion of Chapter 1. There we had argued the for $d = 2, 3$, the (formal) rescaling property of the polymer measure $d\hat{P}_{T,\beta}^{\mathrm{Polym}} = \exp\left[-\beta J_{0,T}\right] dP^{\mathrm{Wiener}}/Z$, which is

$$\hat{P}^{\text{Polym}}_{T,\beta T^{-(4-d)/2}} \rho_T^{-1} = \hat{P}^{\text{Polym}}_{1,\beta},$$

indicates that the self-repellent random walk with coupling parameter $\beta T^{-(4-d)/2}$ should converge, after a Brownian scaling, toward the polymer measure with coupling parameter β. This is the content of Theorem 2.29 for $d = 3$ and has been proved for $d = 2$ by Stoll [69]. The renormalization needed to define the two dimensional polymer measure is just the subtraction of the logarithmically divergent loop diagram, i.e. just by subtracting the expectation. This is an old result of Varadhan [75] who proved (with a different regularization) that

$$Y_T = \lim_{\varepsilon \to 0} \left(J^{\varepsilon}_{0,T} - E J^{\varepsilon}_{0,T} \right)$$

exists in L_2, and $E \exp[-\beta Y_T] < \infty$ for all $\beta \geq 0$. The polymer measure for $d = 2$ is therefore just

$$d\hat{P}^{\text{Polym}}_{T,\beta} = \exp[-\beta Y_T]\, dP^{\text{Wiener}}/Z.$$

Somewhat surprisingly, Y_T has also a positive exponential moment, as has been proved by LeGall:

Proposition 2.31. *([48]) There exists $\beta_o(2) > 0$ such that*

$$E\left(\exp[\beta Y_1]\right) < \infty$$

for $\beta < \beta_o(2)$.

Therefore, the polymer measure exists (for $d = 2$ not for $d = 3$) also with the "wrong sign" if β is small. This makes it plausible that Stoll's result stays correct also in the attractive case. This is in fact true and is the content of the following result by Brydges and Slade:

Theorem 2.32. *Assume $d = 2$. Then there exists $\beta_o(2) > 0$ such that for $0 \leq \beta < \beta_o(2)$*

a) $\sup_{T>0} E^{\text{RW}} \exp[\beta(\gamma_T - E\gamma_T)] < \infty$
c) $\lim_{T\to\infty} \hat{P}_{T,\beta} \rho_T^{-1} = \hat{P}^{\text{Polym}}_{T,-\beta}$

I will not give the details. There are a number of interesting observations:

- The renormalization is necessary. Evidently $E^{\text{RW}} \exp[\beta\gamma_T] \to \infty$ for $\beta > 0$, simply because $\gamma_T \to \infty$ in probability.
- In contrast to the situation for $d \geq 3$, the limit in the two dimensional case depends on β.

2.4 The collapsed phase for self-attracting random walks

In this section, we will discuss some aspects of Theorem 2.27. We will focus on the limiting behavior of $\hat{P}_T l_T^{-1}$, from which actually the other properties can be derived. Therefore, we assume $d = 1$ or $d \geq 2$ and $\beta > \beta_{cr}(d)$. To simplify matters, we will also stick to the case where the variational problem (2.2) has a unique maximizer, which makes things technically a bit simpler. In that case, we can give a more precise version of the theorem, identifying the limit, which we had not been able to do in the case where one has non-uniqueness. For the rest of this section, we therefore assume

Condition 2.33. *$K_\beta = \left\{\theta_x \mu_o : x \in \mathbb{Z}^d\right\}$ for some μ_o.*

This especially applies to $\beta \geq 2d$, but it may well always be true. The proof given here incorporates some technical simplifications compared with the one given in [14] which make it more transparent (I hope).

Theorem 2.34. *Assume $b(\beta) > 0$ and Condition 2.33. Then*

a) $$\lim_{T\to\infty} \hat{P}_T l_T^{-1} = \sum_{x\in\mathbb{Z}^d} \sqrt{\mu_o(-x)}\delta_{\theta_x\mu_o} \Big/ \sum_{x\in\mathbb{Z}^d} \sqrt{\mu_o(x)}$$

b) $$\lim_{T\to\infty} \hat{P}_T X_T^{-1} = \left(\sqrt{\mu_o} * \sqrt{\widetilde{\mu}_o}\right) \Big/ \sum_{x\in\mathbb{Z}^d} \left(\sqrt{\mu_o} * \sqrt{\widetilde{\mu}_o}\right)(x),$$

where $\widetilde{\mu}_o(x) \stackrel{\text{def}}{=} \mu_o(-x)$.

It is also not difficult to determine $\lim_{T\to\infty} \hat{P}_T$ itself which turns out to be a mixture of ergodic Markov processes. This is in spirit very close to Proposition 2.18. I will give some more comments later.

Remark that the choice of μ_o, which is of cause arbitrary in K_β does not influence the right-hand side of the above equations.

The proof of the theorem splits into three parts, which will be presented in the subsections, but I will concentrate on the probabilistic aspects.

A crucial first step is to prove that under $\hat{P}_T$, the local times l_T concentrates with high probability to a neighborhood of K_β. We had coined this the "tube property", because K in the case of uniqueness and $d = 1$ is a sort of an infinite line, so a neighborhood looks like a tube.

On $\mathcal{M}_1^+(\mathbb{Z}^d)$ we take the total variation norm $\|\cdot\|_{\mathrm{TV}}$, which can be taken as the metric for the weak convergence (as $\mathbb{Z}^d$ is countable). If A is a subset of $\mathcal{M}_1^+(\mathbb{Z}^d)$, then we write $U_\varepsilon(A)$ for the ε-neighborhood in total variation of A. So the statement is

Proposition 2.35. *For any $\varepsilon > 0$*

$$\lim_{T\to\infty} \hat{P}_{T,\beta}(l_T \notin U_\varepsilon(K_\beta)) = 0.$$

Remark 2.36. The proof will actually show that for any positive ε, the probability $\hat{P}_{T,\beta}(l_T \notin U_\varepsilon(K_\beta))$ decays exponentially in T.

Even after having proved this "tube property" the path measure could still float around very freely. The next and crucial step is the proof that this does not happen.

Proposition 2.37. *For any $\eta > 0$ there exists $S(\eta) \in \mathbb{N}$ such that for all $\varepsilon > 0$*

$$\limsup_{T\to\infty} \hat{P}_{T,\beta}\left(l_T \notin U_\varepsilon\left(\{\theta_x\mu_o : |x| \le S(\eta)\}\right)\right) \le \eta.$$

From this tightness property, the convergence follows as will explained in subsection 2.4.3.

2.4.1 The tube property: Proof of Proposition 2.35

It is quite evident that Proposition 2.35 should be true, but there is a very annoying problem to prove it. First observe that

$$\hat{P}(l_T \notin U) = \frac{E\left(1_{l_T\notin U}\exp\left[\beta T\,\|l_T\|_2^2\right]\right)}{E\left(\exp\left[\beta T\,\|l_T\|_2^2\right]\right)}.$$

For the estimation of the numerator, we define $F : \mathcal{M}_1^+(\mathbb{Z}^d) \to [-\infty,\infty)$ by $F(\mu) = \|\mu\|_2^2$ if $\mu \notin U$ and $F(\mu) = -\infty$ otherwise. As F is upper semicontinuous, we would expect to get

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\log E\left(1_{l_T\notin U}\exp\left[\beta T\,\|l_T\|_2^2\right]\right) &= \limsup_{T\to\infty}\frac{1}{T}\log E\left(\exp\left[\beta T F(l_T)\right]\right)\\ &\le \sup\{\beta F(\mu) - I(\mu)\}\\ &= \sup_{\mu\notin U}\left\{\beta\,\|\mu\|_2^2 - I(\mu)\right\}\end{aligned}$$

The right hand side of this is evidently strictly smaller than $b(\beta)$, so this would prove the claim. The above inequality is however not quite evident because we only have a weak LDP at our disposal. We can try to remedy the situation by using a compactification argument, i.e. wind the random walk on the torus in the same way as we did in the proof of Proposition 2.23. The problem is that in our situation, the monotonicity argument does not work out such nicely. We would like to argue as follows: Fix some (large) $R \in \mathbb{N}$ and consider again the wound up random walk on the torus $T_R = \{0,\dots,R-1\}^d$. Denote the corresponding set of solutions of the variational problem by $K^R = \left\{\mu \in \mathcal{M}_1^+(T_R) : \beta\,\|\mu\|_2^2 - I^R(\mu) = b^R(\beta)\right\}$. For a given neighborhood U_ε of K we would like to find $\varepsilon' > 0$ such that for any large enough R

$$1_{l_T \notin U_\varepsilon(K)} \exp\left[\beta \left\| l_T \right\|_2^2\right] \leq 1_{l_T^R \notin U_{\varepsilon'}(K^R)} \exp\left[\beta \left\| l_T^R \right\|_2^2\right], \tag{2.15}$$

in which case we would get the desired inequality, by estimating

$$\lim_{T\to\infty} \frac{1}{T} \log E\left(1_{l_T^R \notin U_{\varepsilon'}(K^R)} \exp\left[\beta \left\| l_T^R \right\|_2^2\right]\right) \leq \sup_{\mu \notin U_{\varepsilon'}(K^R)} \left(\beta \left\|\mu\right\|_2^2 - I^R(\mu)\right)$$

which is easily seen to be $< b(\beta)$ if R is large enough. However, (2.15) is not quite correct, as evidently there are probability measures which on the full space are far away from K, but which become close when wound up. On the other hand, it should be clear that such measures must be somewhat weird, and one should be able to control the event where (2.15) fails separately. This is indeed the case.

The proof is based on a reflection trick. Let $i \in \mathbb{N}^+$, $1 \leq m \leq d$ and consider the space of paths D_T (right continuous pure jump) of length T. We define a reflection $\varphi_{m,i}(\omega)$ of paths $\omega \in D_T$ at the hyperplane

$$H_{m,i} \stackrel{\text{def}}{=} \left\{(i_1, \ldots, i_{m-1}, i, i_{m+1}, \ldots, i_d) : (i_1, \ldots, i_{m-1}, i_{m+1}, \ldots, i_d) \in \mathbb{Z}^{d-1}\right\}$$

simply by switching any excursion which moves strictly to the right of the hyperplane to the left. By the "right of the hyperplane", we mean the half-space of points whose m-th coordinates are $> i$, and accordingly for the left. Remark that we start left of the hyperplane as we assume $i > 0$. Therefore, after the switching, the path is at the left of the hyperplane, or on it. It is easy to estimate the density of $P_T \varphi_{m,i}^{-1}$ with respect to P_T. Let $n_{T,i}(\omega)$ be the number of times, the path visits the plane, coming from outside it. Then

Lemma 2.38. $dP_T \varphi_{m,i}^{-1} / dP_T \leq 2^{n_{T,i}}$.

This is fairly evident, and we leave it as an exercise to prove it. The switching costs at most a factor 2 "in entropy" for every visit of the plane (see [14]).

One important and easy property we are using is that "finite size" approximations of the variational problem are approximating the infinite one very well. For a proof of the following Lemma we also refer the reader to [14].

Lemma 2.39. a) $\lim_{R\to\infty} b^R(\beta) = b(\beta)$.

b) *K^R is close to K in the following sense: For any $\varepsilon > 0$ there exists $R_o(\varepsilon)$ such that for any $R \geq R_o$ one has*

b1) *For any $\mu \in K$, the wound up measure μ^R measure on the torus T_R is within $\varepsilon-$distance of some $\nu \in K^R$.*

b2) *For any* $\nu \in K^R$ *one can cut the torus open in such a way (i.e. identify it with the subset* $\{1, \ldots, R\}^d \subset \mathbb{Z}^d$ *such that if* ν *is extended by* 0 *to the whole of* $\mathbb{Z}^d$, *it is within distance less than* ε *to* K.

The Lemma states that analytically, the tube property holds, and we have to prove the probabilistic counterpart. We first state an immediate consequence of the above Lemma 2.39 and the strong LDP on the finite torus.

Lemma 2.40. *Given* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for all* R *large enough*

$$\limsup_{T \to \infty} \frac{1}{T} \log \hat{P}_T \left(l_T^R \notin U_\varepsilon(K^R) \right) \leq -\delta(\varepsilon).$$

Remark that at this stage, no uniformity in R of the estimates for $\hat{P}_T \left(l_T^R \notin U_\varepsilon(K^R) \right)$ is claimed. To prove such an uniformity is essentially the task we have in order to finish the proof of Proposition 2.35.

The idea is as follows: Take $R \gg 1$. Assume we are having a path such that $l_T(\omega)$ is not close to K. We however know from Lemma 2.40 that l_T^R lies with large $\hat{P}_T$-probability close to K^R. By Lemma 2.39, for large enough R, K^R looks much like the translates on the torus of our basic $\mu_o \in K$ (somewhat chopped to fit it onto the torus). Therefore our path, except with very small $\hat{P}_T-$probability, has to distribute its l_T-mass on several essentially disjoint translates of μ_o. The problem is of course that this may happen on an increasing number, with growing T, which looks at first glance difficult to control. Nevertheless, between these chunks of translates of μ_o on which l_T is sitting, there must be vast regions essentially not visited. We select a hyperplane $H_{m,i}$ which is not often visited. Then the reflected path has essentially the same probability as the old one (not quite, of course, but this is measured by Lemma 2.38). As we have enough "empty" space, we can choose the hyperplane in such a way that after the reflection $l_T(\varphi_{m,i}(\omega))$ is *not* close to K^R. Therefore, such a behavior of ω is excluded by Lemma 2.40.

As there are several things which have to tally, we give the details.

If $\delta > 0$ is small enough, we have by our Condition 2.33

$$U_\delta(K) = \bigcup_{l \in \mathbb{Z}^d} U_\delta(\theta_l \mu_o),$$

and similarly on the torus. Of course, we cannot conclude that K^R consists of shifts of one element, and for $\mu \in K$, it will not be true that $\mu^R \in K^R$. However, if $\delta > 0$ is small enough and R sufficiently large, $R \geq R_0(\delta)$, one has

$$U_\delta \left(K^R \right) \subset \bigcup_{k \in T_R} U_{2\delta} \left(\theta_k \mu_o^R \right),$$

as follows immediately from Lemma 2.39. Therefore, if $\varepsilon, \delta > 0$ are small enough we have

$$
\begin{aligned}
&\hat{P}_T\left(l_T^R \in U_\varepsilon(K^R), l_T \notin U_\delta(K)\right) \\
&\leq \hat{P}_T\left(\bigcup_{k\in T_R}\left\{l_T^R \in U_\varepsilon(\theta_k\mu_o^R)\right\}, l_T \notin \bigcup_\ell U_\delta(\theta_\ell\mu_o)\right) \qquad (2.16)\\
&\leq \sum_{k\in T_R}\hat{P}_T\left(l_T^R \in U_\varepsilon(\theta_k\mu_o^R), l_T \notin \bigcup_\ell U_\delta(\theta_\ell\mu_o)\right).
\end{aligned}
$$

We claim that if $\varepsilon \leq \varepsilon_o(\delta)$ (small enough) and $R \geq R_o(\varepsilon,\delta)$ (large enough) then for any

$$\mu \notin \bigcup_\ell U_\delta(\theta_\ell\mu_o) \qquad (2.17)$$

with

$$\mu^R \in U_{2\varepsilon}(\theta_k\mu_o^R) \qquad (2.18)$$

there exists a hyperplane $H_{m,i}$ with $0 < i \leq R$, $1 \leq m \leq d$ such that $(\hat{\mu}_{m,i})^R \notin U_{\delta/4d}(K^R)$, $\left|\|\hat{\mu}_{m,i}\|_2^2 - \|\mu\|_2^2\right| \leq 3\varepsilon$ and $\mu(H_{m,i}) \leq 3\varepsilon$, where $\hat{\mu}_{m,i}$ is the measure where the mass right of the hyperplane $H_{m,i}$ is reflected to the left To see this, remark that for any $\varepsilon > 0$, if (2.18) is satisfied, and R is large enough, then μ must, by Proposition 2.25, be concentrated up to a mass 3ε on R-periodic shifts of boxes of side length $L(\varepsilon)$. Of course, if just one L-box is needed, and ε is small enough compared with δ, then (2.17) cannot be satisfied. Therefore, from this property it follows that for $\varepsilon \leq \varepsilon_o(\delta)$, a single L-box contains at most $1 - 9\delta/10$ of the μ-mass. This holds true uniformly in R (large enough). It is geometrically evident that by choosing $\varepsilon > 0$ small enough and then $R \geq R_o(\varepsilon,\delta)$, we can find a hyperplane $H_{m,i}$ having the following three properties

- $H_{m,i}$ does not intersect any of the L-boxes.
- The L-boxes on the right of $H_{m,i}$ when reflected to the left, do not intersect the L-boxes on the left.
- The right side and the left side of the hyperplane contain at least $\delta/3d$ of the mass of μ.

From these three properties, it is evident that $H_{m,i}$ does the job. Let $A_{m,i}(\varepsilon)$ be the event that the Hamiltonian of the reflected path does not deviate more than 3ε from the unreflected, i.e.

$$A_{m,i}(\varepsilon) \stackrel{\text{def}}{=} \left\{\left|\|l_T \circ \varphi_{m,i}\|_2^2 - \|l_T\|_2^2\right| \leq 3\varepsilon\right\}.$$

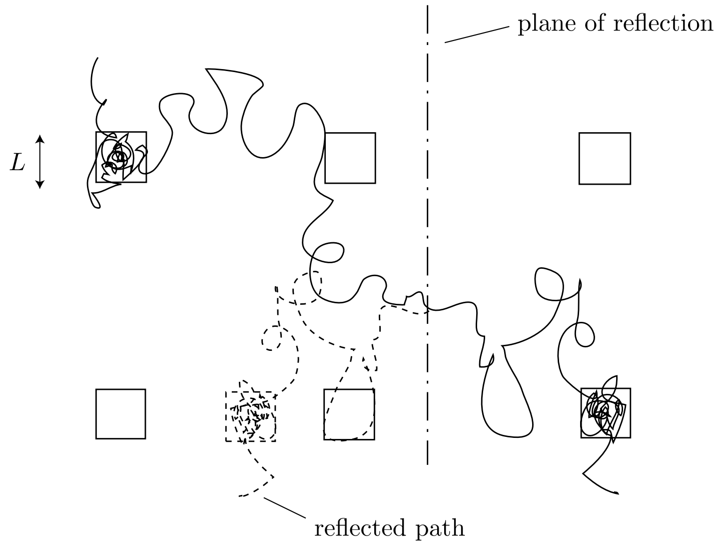


Fig. 2.1.

Then we have

$$\begin{aligned}&\left\{l_T^R \in U_\varepsilon(\theta_k \mu_o^R), l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o)\right\} \\ &\subset \bigcup_{\substack{0<i\leq R\\ 1\leq m\leq d}} \left\{(l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\ A_{m,i}(\varepsilon),\ l_T(H_{m,i}) \leq 3\varepsilon\right\},\end{aligned} \tag{2.19}$$

implying

$$\begin{aligned}&\hat{P}_T\left(l_T^R \in U_\varepsilon(K^R),\ l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o)\right) \\ &\leq R^d \hat{P}_T\left(\bigcup_{\substack{0\leq i\leq R\\ 1\leq m\leq d}} \left\{(l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\ A_{m,i}(\varepsilon),\ l_T(H_i) \leq 3\varepsilon\right\}\right) \\ &\leq R^{d+1} \max_{\substack{0\leq i\leq R\\ 1\leq m\leq d}} \hat{P}_T\left((l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\ A_{m,i}(\varepsilon),\ l_T(H_i) \leq 3\varepsilon\right).\end{aligned} \tag{2.20}$$

We would like to replace the condition $l_T(H_i) \leq 3\varepsilon$ by a condition on $n_{T,i}$. This can be done by still adjusting the ε. By Lemma 2.41 below it follows that for any $\hat{\varepsilon} > 0$ one has for $\varepsilon > 0$ small enough (depending on $\hat{\varepsilon}$)

$$P\left(l_T(H_i) \leq 3\varepsilon,\ n_{T,i} > \hat{\varepsilon}T\right) \leq \exp\left[-(\beta+1)T\right], \tag{2.21}$$

and therefore

$$\hat{P}_T\left(l_T(H_i) \leq \varepsilon,\, n_{T,i} > \hat{\varepsilon}T\right) \leq \exp\left[-T\right].$$

For fixed $\hat{\varepsilon}$, and δ, we can choose $\varepsilon_o(\hat{\varepsilon},\delta)$ such that for $\varepsilon < \varepsilon_o$ the above inequality is true. We can therefore replace the condition $l_T(H_i) \leq 3\varepsilon$ in (2.20) by $n_{T,i} \leq \hat{\varepsilon}T$, making a negligible error. Next, we estimate

$$\begin{aligned}
&\hat{P}_T\left((l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\, A_{m,i}(\varepsilon),\, n_{T,i} \leq \hat{\varepsilon}T\right)\\
&\leq \frac{1}{Z_T} E\left(\mathrm{e}^{\beta T\|l_T\|_2^2}; (l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\, A_{m,i}(\varepsilon),\, n_{T,i} \leq \hat{\varepsilon}T\right)\\
&\leq \frac{\mathrm{e}^{3\beta\varepsilon T}}{Z_T} E\left(\mathrm{e}^{\beta T\|l_T\circ\varphi_i\|_2^2};\, (l_T \circ \varphi_{m,i})^R \notin U_{\delta/4d}(K^R),\, n_{T,i} \leq \hat{\varepsilon}T\right)\\
&\leq \frac{\mathrm{e}^{3\beta\varepsilon T}}{Z_T} E\left(2^{n_{T,i}}\mathrm{e}^{\beta T\|l_T\|_2^2};\, l_T^R \notin U_{\delta/4d}(K^R),\, n_{T,i} \leq \hat{\varepsilon}T\right)\\
&\leq \mathrm{e}^{4\beta\varepsilon T} 2^{\hat{\varepsilon}T}\hat{P}_T\left(l_T^R \notin U_{\delta/2}(K^R)\right).
\end{aligned}$$

Therefore, for given $\beta,\delta > 0$ we choose $\hat{\varepsilon}$ small enough such that the decay of $\hat{P}_T\left(l_T^R \notin U_{\delta/4d}(K^R)\right)$ which is guaranteed by Lemma 2.40 beats $\mathrm{e}^{4\beta\varepsilon T}2^{\hat{\varepsilon}T}$, and then for $\varepsilon \leq \varepsilon_o(\beta,\delta)$, and then R large enough, one gets the desired estimate for $\hat{P}_T\left(l_T \notin U_\delta(K)\right)$, which finishes the proof of Proposition 2.35.

Lemma 2.41. *If* ζ_i, $i \geq 1$, *is a sequence of exponentially distributed random variables, with parameter* 1, *then for* $t \leq 1$

$$P\left(\sum_{i=1}^n \zeta_i \leq nt\right) \leq \exp[-nh(t)],$$

where

$$\lim_{t\to 0} h(t) = -\infty.$$

Proof. This is the standard one dimensional large deviation estimate. The rate function is

$$h(t) = \sup_{\lambda\leq 0}\left(\lambda t - \log\int_0^\infty \exp[\lambda x - x]dx\right) = t - 1 + \log\frac{1}{t}.$$

2.4.2 Tightness: Proof of Proposition 2.37

This is the crucial step of the whole argument. It should be noted that the different places in K_β cannot be distinguished on a logarithmic scale. In fact, it is rather evident that

$$\lim_{T\to\infty}\frac{\log E\left(\exp\left[\beta T\left\|l_T\right\|_2^2\right];\, l_T \in U_\varepsilon(\mu_x)\right)}{\log Z_{T,\beta}} = 1$$

for all $x \in \mathbb{Z}^d$, $\varepsilon > 0$. The most natural way to proceed would be an evaluation of the expression $P\left(l_T \in U_\varepsilon(\mu_x)\right)$ up to a factor $(1 + o(1))$, and the same for $Z_{T,\beta}$. In that case we could control the quotients. This would probably be possible but has not been done in the present context, and is probably technically delicate. We instead make use of the symmetry properties.

Before we start with the formal proof, I want to explain shortly the main idea. We want to conclude that under $\hat{P}_T$, there is only a small probability that $l_T \sim \mu_x$, x far away from 0. To do this, it suffices to get estimates for large $|x|$ of

$$\frac{\hat{P}_T\left(l_T \sim \mu_x\right)}{\hat{P}_T\left(l_T \sim \mu_o\right)} = \frac{E\left(\exp\left[\beta T \left\|l_T\right\|_2^2\right]; \, l_T \sim \mu_x\right)}{E\left(\exp\left[\beta T \left\|l_T\right\|_2^2\right]; \, l_T \sim \mu_o\right)},$$

of course only in the $T \to \infty$ limit. As already remarked, a way to control this would be to evaluate the numerator and the denominator up to a factor $(1 + o(1))$, as $T \to \infty$. There is however a much easier way to get such a control, which in spirit resembles the Peierls argument for the Ising model. If $l_T \sim \mu_x$, $|x|$ large, then the path spends most of the time near x. It is not difficult to prove that the path has to reach the neighborhood of x relatively quickly, say after time $t \ll T$, where however t has to become large when $|x|$ is large. We are doing now a splitting of the local time in the part before time t and after that. If we calculate $\left\|l_T\right\|_2^2$, we now want to split this, too, and we use that the path before t has not much intersection with the path after t, the latter hanging around x, whereas the former doesn't. We compare that now with the situation where the path after t would hang around 0 instead of around x. We just shift any path after time t which hangs around x by $-x$. Of course, we have to fit the path before t to this situation.

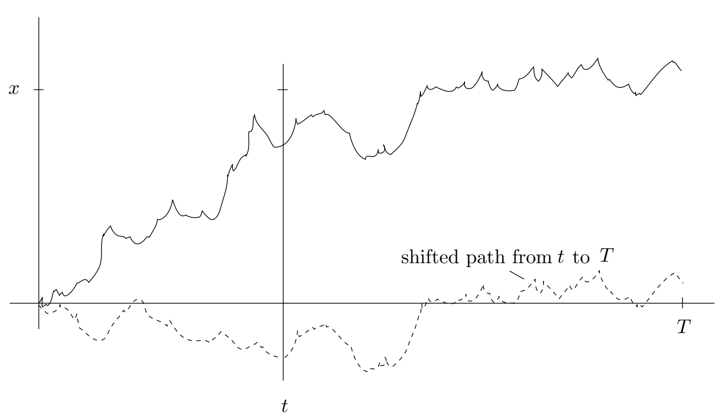


Fig. 2.2.

The crucial point is that the shifting of the main part, namely the path in $[t, T]$ does not cost anything, due to the shift invariance which is used heavily. Lets denote the local time after the shift by l_T^{shift}. The only thing which essentially distinguishes $\|l_T\|_2^2$ from $\left\|l_T^{\mathrm{shift}}\right\|_2^2$ is that the former has essentially no contribution from the interaction between the path on $[0, t]$ with the path on $[t, T]$, where the latter has. It turns out that typically

$$\left\|l_T^{\mathrm{shift}}\right\|_2^2 \approx \|l_T\|_2^2 + Ct.$$

Therefore, we get

$$E\left(\exp\left[\beta T\,\|l_T\|_2^2\right];\, l_T \sim \mu_x\right) \lessapprox \mathrm{e}^{-Ct}\, E\left(\exp\left[\beta T\,\|l_T\|_2^2\right];\, l_T \sim \mu_0\right),$$

uniformly in T large. On the other hand, $|x|$ cannot be large without t, the first time, the path reaches the neighborhood of x, is not also large. Therefore, it follows, that $l_T \sim \mu_x$ can happen only with small $\widehat{P}_T$ probability, uniformly in T for T large. Of course, this is very hand-waiving, and we now give the details of the argument.

We are proving a superficially weaker result than Proposition 2.37:

Proposition 2.42. *There exists $\varepsilon_o > 0$ such that for all $\varepsilon \le \varepsilon_o, \eta > 0$ there exist $S(\varepsilon, \eta) \in \mathbb{N}$, $T_o(\varepsilon, \eta) > 0$ with*

$$\hat{P}_T\left(\bigcup_{|x| \ge S(\varepsilon,\eta)} \{l_T \in U_\varepsilon(\mu_x)\}\right) \le \eta,$$

for $T \ge T_o(\varepsilon, \eta)$, where $\mu_x \overset{\mathrm{def}}{=} \theta_x \mu_o$.

Together with the Proposition 2.35, this evidently implies the Proposition 2.37.

Perhaps some comments on the rôle of ε in our proof is in order, in particular as this point had been not so well handled in the original paper [14]. We will on several occasions switch from one ε to another. As the tube property 2.35 states that outside any of any ε-neighborhoods of K_β there is only (exponentially) negligible $\hat{P}_T$-mass, we can do this freely, always after having chosen T large enough. As the $T \to \infty$ limit will always be the first to perform, this causes no problem.

If $r \in \mathbb{N}$, let $C_r \overset{\mathrm{def}}{=} \{-r, -r+1, \ldots, r\}^d$ and for $x \in \mathbb{Z}^d$, $C_r(x) = C_r + x$. $\partial C_r(x)$ is the inner boundary, i.e.

$$\partial C_r(x) = \{y \in C_r(x) : |y_i - x_i| = r \text{ for some } i\}.$$

We denote by $\tau_r(x)$ the first hitting time of $\partial C_r(x)$ and by $\xi_r(x)$ the time the process spends on $\partial C_r(x)$ after $\tau_r(x)$ before leaving it for the first time. We need some control that the process does not leave $\partial C_r(x)$ too quickly.

Lemma 2.43.

$$\lim_{\rho\to 0} \hat{P}_T\left(\xi_r(x) \le \rho\right) = 0,$$

uniformly in T, r, x.

Proof. Define

$$Y_t(\omega) = \begin{cases} \omega_t & \text{for } t \le \tau \\ \omega_{t+\xi} & \text{for } t > \tau \end{cases}.$$

Then $\{Y_t\}_{t>0}$ and ξ_r are independent under P. Let $l'_T(x) = (1/T)\int_0^T 1_{\{Y_s=x\}}$. Then

$$\left| \|l_T\|_2^2 - \|l'_T\|_2^2 \right| \le C\frac{\xi}{T}. \tag{2.22}$$

Therefore,

$$\begin{aligned} E\left(\exp\left[\beta T \|l_T\|_2^2\right]; \xi \le \rho\right) &\le C\beta E\left(\exp\left[\beta T \|l'_T\|_2^2\right]; \xi \le \rho\right) \\ &\le C\beta\rho E\left(\exp\left[\beta T \|l'_T\|_2^2\right]; \xi \le 1\right) \\ &\le C\beta\rho E\left(\exp\left[\beta T \|l_T\|_2^2\right]\right). \end{aligned}$$

This proves the claim.

We need a further technical Lemma:

Lemma 2.44. *Given $\eta > 0$, there exists $r_o(\eta)$ such that*

$$\sup_{T\ge 1} \sum_{r=r_o(\eta)}^{\infty} \hat{P}_T\left(\tau_r(0) \le \sqrt{r}\right) \le \eta.$$

Proof. We introduce for $t < T$:

$$l_{t,T}(y) \stackrel{\text{def}}{=} \frac{1}{T-t}\int_t^T 1_{X_u=y}\, du.$$

Then, as above in (2.22), we have

$$\left| T\|l_T\|_2^2 - T\left\|l_{\sqrt{r},T+\sqrt{r}}\right\|_2^2 \right| \le C\sqrt{r}.$$

Therefore

$$\begin{aligned} E\left(\exp[T\beta\|l_T\|_2^2]; \tau_r \le \sqrt{r}\right) &\le \mathrm{e}^{C\sqrt{r}} E\left(\exp[T\beta\left\|l_{\sqrt{r},T+\sqrt{r}}\right\|_2^2]\right) P\left(\tau_r \le \sqrt{r}\right) \\ &= \mathrm{e}^{C\sqrt{r}} Z_{T,\beta} P\left(\tau_r \le \sqrt{r}\right). \end{aligned}$$

If the random walk reaches $C_r(0)$ in time $\le \sqrt{r}$, it has to make at least r jumps in this time. Applying now Lemma 2.41, the claim follows.

Given ε, let $\ell(\varepsilon)$ be chosen such that $\mu_o(C^c_{\ell(\varepsilon)}) \leq \varepsilon$. The main step in the proof of Proposition 2.42 is given by the following result:

Lemma 2.45. *If $\varepsilon > 0$ is small enough, there exists $A(\varepsilon) > 0$, such that for x with $|x|, T, u \geq A(\varepsilon)$ and $\rho \in (0,1]$*

$$\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x), \xi_{\ell(\varepsilon)}(x) > \rho, \tau_{\ell(\varepsilon)}(x) > u\right) \leq \frac{C}{\rho} \exp\left[-u/C\right], \tag{2.23}$$

(C may depend on β and d but on nothing else, as usual).

Proof. We abbreviate $\xi_{\ell(\varepsilon)}(x)$ as ξ, and $\tau_{\ell(\varepsilon)}(x)$ as τ during this proof. $A(\varepsilon)$ is chosen in any case bigger than $\ell(\varepsilon)$. Then 0 is outside $B_{\ell(\varepsilon)}(x)$. Remark first that for $|x| > \ell(\varepsilon)$ and $l_T \in U_\varepsilon(\mu_x)$, the process can spend outside of $C_{\ell(\varepsilon)}(x)$ only a total time less than a proportion of εT. Therefore, on $\{l_T \in U_\varepsilon(\mu_x)\}$ we have

$$\tau \leq c_1 \varepsilon T \leq T$$

if ε is small enough. (We remind the reader that $c_1, c_2, \ldots$ are constants not changed after having them introduced). Therefore

$$\begin{aligned}
&E\left(\mathrm{e}^{\beta T \|l_T\|_2^2};\, l_T \in U_\varepsilon(\mu_x),\, \xi > \rho,\, u < \tau\right) \\
&\leq E\left(\mathrm{e}^{\beta T \|l_T\|_2^2};\, l_T \in U_\varepsilon(\mu_x),\, \xi > \rho,\, u < \tau \leq c_1 \varepsilon T.\right) \\
&\leq \frac{1}{\rho} \int_u^{c_1 \varepsilon T + \rho} dt\, E\left(\mathrm{e}^{\beta T \|l_T\|_2^2};\, l_T \in U_\varepsilon(\mu_x),\, \xi > \rho,\, t - \rho < \tau \leq t\right) \\
&\leq \frac{1}{\rho} \int_u^{c_1 \varepsilon T + 1} dt\, E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x),\, X_t \in \partial C_{\ell(\varepsilon)}(x),\, t - 1 < \tau\right),
\end{aligned} \tag{2.24}$$

where in the last inequality we have used that on the set $\{t - \rho < \tau \leq t, \xi > \rho\}$ we have $X_t \in \partial C_{\ell(\varepsilon)}(x)$. We can assume that $c_1 \varepsilon T + 1 \leq T$, so that $t \leq T$ in the domain of integration.

We have the convex combination $l_T = \frac{t}{T} l_t + \frac{T-t}{T} l_{t,T}$, and therefore

$$T \|l_T\|_2^2 = \frac{t^2}{T} \|l_t\|_2^2 + \frac{(T-t)^2}{T} \|l_{t,T}\|_2^2 + 2\frac{t(T-t)}{T} \langle l_t, l_{t,T} \rangle . \tag{2.25}$$

If $A(\varepsilon) \geq 1/\varepsilon$, we have $\varepsilon T \leq 1$ for $T \geq A(\varepsilon)$ and therefore $t \leq C\varepsilon T$ if $t \leq c_1 \varepsilon T + 1$. Therefore

$$\frac{t^2}{T} \|l_t\|_2^2 \leq C \varepsilon T, \tag{2.26}$$

if $\varepsilon \leq 1$, which we of cause assume. We now estimate the third summand in (2.25). First observe that

$$\|l_{t,T} - l_T\|_{\mathrm{TV}} \leq t/T \leq C\varepsilon,$$

and therefore

$$l_{t,T} \in U_{c_2\varepsilon}(\mu_x) \tag{2.27}$$

if $l_T \in U_\varepsilon(\mu_x)$, and we can conclude

$$|\langle l_t, l_{t,T}\rangle| \leq |\langle l_t, \mu_x\rangle| + C\varepsilon.$$

Now, l_{t-1} does not charge $C_{l(\varepsilon)}(x)$ if $t-1<\tau$, and as $\mu_x\left(C_{l(\varepsilon)}(x)^c\right) \leq \varepsilon$, we conclude

$$|\langle l_t, \mu_x\rangle| \leq |\langle l_{t-1}, \mu_x\rangle| + \|l_{t-1} - l_t\|_{\mathrm{TV}} \leq C\varepsilon,$$

if $t \geq u \geq A(\varepsilon)$, $A(\varepsilon) \geq 1/\varepsilon$. Therefore, on $\{l_T \in U_\varepsilon(\mu_x), t-1 \leq \tau\}$, and the region we are considering, we have

$$\left|2\frac{t(T-t)}{T}\langle l_t, l_{t,T}\rangle\right| \leq C\varepsilon \tag{2.28}$$

Implementing (2.26), (2.28) and (2.27) into (2.24), we get

$$\begin{aligned} &E\left(\mathrm{e}^{\beta T\|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi > \rho, u < \tau\right) \qquad (2.29)\\ &\leq \frac{1}{\rho}\int_u^{c_1\varepsilon T+1} dt\, \mathrm{e}^{C\varepsilon\beta t} E\left(\mathrm{e}^{\beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2}; l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t \in \partial C_{\ell(\varepsilon)}(x)\right). \end{aligned}$$

We next claim that for $y \in \partial C_{\ell(\varepsilon)}(x)$

$$E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right) \geq \exp\left[c_3 t + \beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right] \tag{2.30}$$

on $\{l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t = y\}$, where $\mathcal{F}_{t,T}$ is the σ-field generated by X_s, $t \leq s \leq T$. Before proving this, we show that (2.29) and (2.30) imply the Proposition 2.42.

$$\begin{aligned} Z_{T,\beta} &= E_x\left(E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right)\right)\\ &\geq \sum_{y\in\partial C_{\ell(\varepsilon)}(x)} E_x\left(E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right); l_{t,T} \in U_{c_2\varepsilon}(\mu_x),\ X_t = y\right)\\ &\geq \sum_{y\in\partial C_{\ell(\varepsilon)}(x)} \frac{p_t(x,y)}{p_t(0,y)} E\left(E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right); l_{t,T} \in U_{c_2\varepsilon}(\mu_x),\ X_t = y\right). \end{aligned}$$

Remark that if $A(\varepsilon)$ is large enough, we have $p_t(x,y)/p_t(0,y) \geq 1$ for all $y \in \partial C_{\ell(\varepsilon)}(x)$ and $t \geq A(\varepsilon)$, if $|x| \geq A(\varepsilon)$. Therefore,

$$\begin{aligned} Z_{T,\beta} &\geq \sum_{y\in\partial C_{\ell(\varepsilon)}(x)} E\left(\exp\left[c_3 t + \beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right]; l_{t,T} \in U_{c_2\varepsilon}(\mu_x),\ X_t = y\right)\\ &= \mathrm{e}^{c_3 t} E\left(\exp\left[\beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right]; l_{t,T} \in U_{c_2\varepsilon}(\mu_x),\ X_t \in \partial C_{\ell(\varepsilon)}(x)\right). \end{aligned}$$

Therefore, by (2.29),

$$E\left(\mathrm{e}^{\beta T\|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi_{r_\varepsilon(x)} > \rho, u < \tau\right)$$
$$\leq \frac{Z_{T,\beta}}{\rho} \int_u^{c_1\varepsilon T+1} \mathrm{e}^{-c_3 t + C\varepsilon\beta t}\, dt \leq C\frac{Z_{T,\beta}}{\rho}\, \mathrm{e}^{-Cu},$$

provided $\varepsilon \leq \varepsilon_0(\beta)$. This proves Proposition 2.42.

It remains to prove (2.30). On the prescribed event, the left hand side of (2.30) is

$$\geq \exp\left[\beta\frac{(T-t)^2}{T}\,\|l_{t,T}\|_2^2\right] E_x\left(\mathrm{e}^{2t\beta\langle l_t,\mu_x\rangle}\,\Big|\, X_t = y\right)\mathrm{e}^{-C\varepsilon t}.$$

We make a transformation of the path measure switching to the measure $P_x^{(\mu_x)}$ of a Markov process starting in x having Q-matrix

$$\left(\frac{1}{2}\sqrt{\mu_x(j)/\mu_x(j)}\right)_{i,j\in\mathbb{Z}^d}.$$

P_x is absolutely continuous on $(D_t, \mathcal{F}_t)$ with respect to $P_x^{(\mu_x)}$ with a density

$$\frac{dP_x}{dP_x^{(\mu_x)}}(\omega) = \sqrt{\frac{\mu_x(x)}{\mu_x(\omega_t)}}\exp\left[\int_0^t \frac{\frac{1}{2}\Delta\sqrt{\mu_x(\omega_s)}}{\sqrt{\mu_x(\omega_s)}}\, ds\right], \tag{2.31}$$

where Δ is the discrete Laplacian $\Delta f(x) = \sum_{y:|y-x|=1}(f(y)-f(x))$. (see e.g. [63], Chapter IV.3). We write now $\mu_o(x) = g^2(x)$. g satisfies the Euler equation

$$4\beta g(x)^3 + \Delta g(x) = \lambda g(x). \tag{2.32}$$

Multiplying with $g(x)$ and summing over x gives

$$\lambda = 4\beta\sum_x g\,(y)^4 - 2I(g^2) \geq 2b(\beta) > 0.$$

On the other hand, if we divide 2.32 by $g(x)$, we get

$$\int_0^t \frac{\frac{1}{2}\Delta\sqrt{\mu_o(\omega_s)}}{\sqrt{\mu_o(\omega_s)}}\, ds + 2\beta\,\langle l_t, \mu_o\rangle = \lambda.$$

The same is of course true if we replace μ_o by μ_x. Implementing this into (2.31), implies

$$E_x\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle}\,\Big|\, X_t = y\right) \geq E_x\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle}; X_t = y\right) \tag{2.33}$$

$$= \mathrm{e}^{\lambda t/2} \sqrt{\frac{\mu_o(0)}{\mu_o(y-x)}} P_x^{(\mu_x)}(X_t = y).$$

$(X_t)_{t \geq 0}$ under $P_x^{(\mu_x)}$ is ergodic with stationary measure μ_x. Therefore

$$\lim_{t \to \infty} P_x^{(\mu_x)}(X_t = y) = \mu_x(y).$$

Therefore, there exists $t_o(\varepsilon) > 0$ such that for $t \geq t_o(\varepsilon)$, and all $y \in \partial C_{\ell(\varepsilon)}(x)$

$$E_x \left(\mathrm{e}^{2t \langle l_t, \mu_x \rangle} \middle| X_t = y \right) \geq \mathrm{e}^{\lambda t/3},$$

Therefore, we only have to choose $A(\varepsilon) \geq t_o(\varepsilon)$. This proves (2.30).

The Lemmas 2.43, 2.44 and 2.45 now imply Proposition 2.42 in the following way. Given $\eta > 0$, we first choose $\rho(\eta) > 0$ according to Lemma 2.43 such that

$$\hat{P}_T(\xi_r(x) \leq \rho(\eta)) \leq \eta/3 \tag{2.34}$$

for all T, x, r. Then we choose $r_o(\eta/3)$ according to Lemma 2.44, so that

$$\hat{P}_T\left(\tau_r(0) \leq \sqrt{r} \,\text{for some}\, r \geq r_o(\eta/3)\right) \leq \eta/3. \tag{2.35}$$

If $\varepsilon > 0$ is given, we choose $u(\varepsilon, \eta) \in \mathbb{N}$, $u(\varepsilon, \eta) \geq \max\left(A(\varepsilon), r_o(\eta/3)^2\right)$ such that

$$\frac{C}{\rho(\eta)} \exp[-u(\varepsilon, \eta)/C] \leq \eta/3.$$

(C here from Lemma 2.45). If now

$$|x| \geq S(\varepsilon, \eta) \stackrel{\text{def}}{=} \max\left(A(\varepsilon) + l(\varepsilon), u(\varepsilon, \eta)^2\right)$$

then on the complement of the event in (2.35), one has $\tau_{l(\varepsilon)}(x) \geq \tau_{u(\varepsilon,\eta)^2}(0)$, and therefore, according to Lemma 2.45, we have

$$\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x), \xi_{\ell(\varepsilon)}(x) > \rho(\eta), \tau_{\ell(\varepsilon)}(x) > u(\varepsilon, \eta)\right) \leq \eta/3. \tag{2.36}$$

Combining (2.34),(2.35) and (2.36) we get the statement of Proposition 2.42.

In the next section we will need a result which can be proved by a extension of the above argument:

Proposition 2.46. $\lim_{m \to \infty} \limsup_{T \to \infty} \sup_{0 \leq t \leq T} \hat{P}_T(|X_t| \geq m) = 0$.

We will not give a proof here which is essentially a repetition of the arguments above. Remark that the supremum over t is outside. The proof of Proposition 2.37 given above shows that l_T must have its main weight, up to small probability, close to the starting point. The argument essentially is that if it would be far out, then the path would need some time to reach this place, which would be bad for the Hamiltonian. Completely similar arguments show that uniformly in t we can estimate the probability that the path is at time t far out, just because the path would have to go out from the main bulk out to this place and come back, which again would be bad for the Hamiltonian.

2.4.3 Proof of Theorem 2.34

For small enough $\varepsilon > 0$, the $U_\varepsilon(\mu_x)$, $x \in \mathbb{Z}^d$, are all disjoint. We also know from Proposition 2.37 and Proposition 2.46 that for any $\eta > 0$ there exist $S(\eta)$ and $m(\eta) > 0$ (not depending on ε !) such that for $T \geq T_o(\varepsilon, \eta)$ one has

$$\sup_{t \leq T} \hat{P}_T\left(l_T \in \bigcup_{x:|x| \leq S(\eta)} U_\varepsilon(\mu_x), |X_t| \leq m(\eta) \right) \geq 1 - \eta.$$

Of course, we can assume $S(\eta) \leq m(\eta)$. We want to prove that for all $x \in \mathbb{Z}^d$

$$\lim_{T \to \infty} \hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) = \frac{\sqrt{\mu_o(-x)}}{\sum_y \sqrt{\mu_o(y)}}, \tag{2.37}$$

for all small enough ε, i.e. $\varepsilon \leq \varepsilon_o$, ε_o depending on nothing except d and β. Remark that $\mu_o(-x) = \mu_x(0)$. From Proposition 2.37 we see that in order to prove (2.37) it suffices to prove that for any x, we have

$$\lim_{T \to \infty} \frac{\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right)}{\hat{P}_T\left(l_T \in U_\varepsilon(\mu_o)\right)} = \frac{\sqrt{\mu_x(0)}}{\sqrt{\mu_o(0)}} \tag{2.38}$$

Given Proposition 2.35, this proves Theorem 2.34, part a). We therefore fix x and take $\eta > 0$ such that $|x| \leq S(\eta) \leq m(\eta)$. We will later on choose some $t = t(\eta)$, which will *not* depend on ε (provided always that ε is small enough). We use again the splitting

$$T \left\| l_T \right\|_2^2 = \frac{t^2}{T} \left\| l_t \right\|_2^2 + \frac{(T-t)^2}{T} \left\| l_{t,T} \right\|_2^2 + 2\frac{t(T-t)}{T} \left\langle l_t, l_{t,T} \right\rangle .$$

If $T_o(\varepsilon, \eta)$ is chosen large enough (after $t = t(\eta)$ is chosen), we have the first summand $O(\eta)$ and replacing the numerator $T - t$ in the third summand by T causes an error of the same size. Therefore,

$$\begin{aligned} &\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) \\ &= \hat{P}_T\left(l_T \in U_\varepsilon(\mu_x),\, |X_t| \leq m(\eta)\right) + O(\eta) \qquad (2.39) \\ &= \frac{1}{Z_T} E\left(\mathrm{e}^{2t\langle l_t, l_{t,T}\rangle + \frac{(T-t)^2}{T}\|l_{t,T}\|_2^2};\, l_T \in U_\varepsilon(\mu_x),\, |X_t| \leq m(\eta) \right) + O(\eta). \end{aligned}$$

We want to make some further replacements. Remark that if $T \geq T_o(\varepsilon, \eta)$ and $T_o(\varepsilon, \eta)$ is appropriate, we have $\left\{l_T \in U_{\varepsilon/2}(\mu_x)\right\} \subset \left\{l_{t,T} \in U_\varepsilon(\mu_x)\right\} \subset \left\{l_T \in U_{2\varepsilon}(\mu_x)\right\}$, and by our tube property, we get

$$\lim_{T \to \infty} \left[\hat{P}_T\left(l_T \in U_{2\varepsilon}(\mu_x)\right) - \hat{P}_T\left(l_T \in U_{\varepsilon/2}(\mu_x)\right) \right] = 0.$$

Therefore, we can replace $l_T \in U_\varepsilon(\mu_x)$ in (2.39) by $l_{t,T} \in U_\varepsilon(\mu_x)$, making an additional error $O(\eta)$. Next, we replace $l_{t,T}$ in $\langle l_t, l_{t,T}\rangle$ by μ_x, which causes an error in the exponent of order $t\varepsilon$. Summarizing, we get for $\varepsilon t \leq 1$

$$
\begin{aligned}
&\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) \\
&\qquad = \frac{1}{Z_T} E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle + \frac{(T-t)^2}{T}\|l_{t,T}\|_2^2};\ l_{t,T} \in U_\varepsilon(\mu_x),\ |X_t| \le m(\eta)\right) \\
&\qquad\quad \times \left(1 + O\left(\varepsilon t\right)\right) + O(\eta) \\
&\qquad = \frac{1}{Z_T} \sum_{y:|y|\le m} E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle + \frac{(T-t)^2}{T}\|l_{t,T}\|_2^2};\ l_T \in U_\varepsilon(\mu_x),\ X_t = y\right) \\
&\qquad\quad \times \left(1 + O\left(\varepsilon t\right)\right) + O(\eta) \\
&\qquad = \frac{1}{Z_T} \sum_{y:|y|\le m} E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle} \,\middle|\, X_t = y\right) \\
&\qquad\quad \times E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_x)\right)\left(1 + O(\varepsilon t)\right) + O(\eta).
\end{aligned}
$$

The crucial point is that we can choose ε depending on η, and we do it in such a way that $\varepsilon t \le \eta$, of course, after having chosen $t\left(\eta\right)$, which will be done below. In this way, we can replace the error terms above by a summand $O\left(\eta\right)$, but we have then to take $T \ge T_o\left(\eta, \varepsilon\right)$. We explain now, how $t\left(\eta\right)$ has to be chosen. We use the same transformation as in the last section (see 2.33) and get

$$
E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle} \,\middle|\, X_t = y\right) = \frac{\mathrm{e}^{\lambda t/2}\sqrt{\frac{\mu_x(0)}{\mu_x(y)}}P_0^{(\mu_x)}\left(X_t = y\right)}{P_x\left(X_t = y\right)}.
$$

If we let $t \to \infty$, we get uniformly in $|y| \le 2m\left(\eta\right)$, $P_x\left(X_t = y\right) = Ct^{-d/2}\left(1 + o\left(1\right)\right)$, and $P_0^{(\mu_x)}\left(X_t = y\right) = \mu_x\left(y\right)\left(1 + o\left(1\right)\right)$. Therefore, if we put $\phi\left(t\right) \stackrel{\mathrm{def}}{=} \mathrm{e}^{\lambda t/2} t^{d/2}/C$, we get for $t \ge t_o\left(\eta, m\right)$

$$
E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle} \,\middle|\, X_t = y\right) = \phi(t)\sqrt{\mu_x(0)}\sqrt{\mu_x(y)}\left(1 + O\left(\eta\right)\right),
$$

uniformly in $|y| \le 2m\left(\eta\right)$. Therefore, after having chosen $m\left(\eta\right)$, we choose $t\left(\eta\right)$ in this way, and then we have

$$
\begin{aligned}
&\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) \\
&\le \frac{\phi\left(t\right)}{Z_T}\sqrt{\mu_x(0)} \sum_{y:|y|\le m(\eta)} \sqrt{\mu_x(y)} E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_x)\right) + O(\eta).
\end{aligned}
$$

Remark now, that $\mu_x(y) = \mu_o\left(y - x\right)$ and

$$
E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_x)\right) = E_{y-x}\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_o)\right).
$$

As $|x| \le m\left(\eta\right)$, we therefore get for $T \ge T_o\left(\eta, \varepsilon\left(\eta\right)\right)$:

$$
\begin{aligned}
&\hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) \\
&\leq \frac{\phi(t)}{Z_T}\sqrt{\mu_x(0)} \sum_{y:|y|\leq 2m} \sqrt{\mu_o(y)} E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_o)\right) + O(\eta) \\
&\leq \frac{1}{Z_T}\frac{\sqrt{\mu_x(0)}}{\sqrt{\mu_o(0)}} \sum_{y:|y|\leq 2m} \sqrt{\mu_o(0)}\sqrt{\mu_o(y)} \\
&\quad \times E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_{T-t} \in U_\varepsilon(\mu_o)\right) + O(\eta) \\
&\leq \frac{\sqrt{\mu_x(0)}}{\sqrt{\mu_o(0)}}\hat{P}_T\left(l_T \in U_\varepsilon(\mu_o),\, |X_t| \leq 2m(\eta)\right) + O(\eta) \\
&\leq \frac{\sqrt{\mu_x(0)}}{\sqrt{\mu_o(0)}}\hat{P}_T\left(l_T \in U_\varepsilon(\mu_o)\right) + O(\eta).
\end{aligned}
$$

The above conclusion is for $\varepsilon = \varepsilon(\eta)$, where the latter has been chosen above, but thanks to Proposition 2.35, we can switch back to a fixed $\varepsilon > 0$, small enough, but not depending on η, if T is large enough. In conclusion, we get for a fixed (small) ε

$$
\limsup_{T\to\infty} \hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) \leq \frac{\sqrt{\mu_x(0)}}{\sqrt{\mu_o(0)}} \liminf_{T\to\infty} \hat{P}_T\left(l_T \in U_\varepsilon(\mu_o)\right).
$$

As the role of x and 0 are interchangeable in the argument, we get the desired relation (2.38). Therefore, part a) of the Theorem 2.34.

I will not give the details of part b), as it is by some straightforward modification and extension of the above argument. One has only to introduce another splitting at a time point $T-t$, to "separated" the endpoint from the main bulk of the empirical distribution.

Remark 2.47. From the above proof, it is not difficult to guess what $\lim_{T\to\infty} \hat{P}_T$ is, still assuming uniqueness modulo shifts: It is just a mixing of the Markovian processes $P_0^{(\mu_x)}$ which are the jump processes with Q-matrix

$$
\left(\frac{1}{2}\sqrt{\mu_x(j)/\mu_x(i)}\right)_{i,j\in\mathbb{Z}^d,\, i\neq j}.
$$

The mixing is over x and is given by $\sqrt{\mu_x(0)}\Big/\sum_y \sqrt{\mu_y(0)}$. Therefore, the limiting measure is not Markovian itself.

Remark 2.48. Some last remark about what happens if the Condition 2.33 would fail (a case where I don't know if it occurs at all). In that case it would be difficult to establish a limiting result and one would have to go into finer asymptotics in large deviation in order to determine the relative weights on the different fibres. This has not been done for the present problem (see however [13] for the case of sums of i.i.d. random vectors). However, one can easily get some information: The proof of the tightness essentially applies with only small modifications, and one gets at least tightness for instance of the distribution of the endpoint (and the relative distribution inside each fibre of the K_β) without any further assumptions besides $b(\beta) > 0$. For the details, I refer to [14].

2.5 A droplet construction for the Wiener sausage

A problem which is closely related to the one in the previous section is connected with the classical large deviation result of Donsker and Varadhan for the volume of the Wiener sausage [34]. There is a corresponding result for random walks where the volume of the Wiener sausage is replaced by the number of points visited. I will sketch some of the problems and results in this section without going into technical details.

I stick for the moment to the Wiener sausage: So let β_t, $t > 0$, be the standard Brownian motion on $\mathbb{R}^d$, starting in 0. The Wiener sausage is defined by

$$W_T^a = \bigcup_{s \leq T} B_a(\beta_s),$$

where $a > 0$ and where $B_a(x)$ is the ball with radius a and center x. All results generalize also to the situation where $B_a(x)$ is replace by $x + C$ where C is an arbitrary compact set of positive capacity. The volume of the Wiener sausage is then just its Lebesgue measure

$$V_T^a = |W_T^a|\,.$$

For a random walk (discrete time, say), the natural quantity is the number of points N_T visited by the random walk up to time T. It is known that for $d \geq 3$, $EV_T^a = \kappa_a T + o\,(T)$, where κ_a is the Newtonian capacity of B_a (see [67]). For $d = 2$, one has $EV_T^a \sim T/\log T$, and for $d = 1$, it is of course of order $\sqrt{T}$. One is then interested in estimating the probability that V_T^a is smaller than that. A possibility to measure that is to investigate $E\,(\exp[-\beta V_T^a])$, $\beta > 0$.

This quantity appears in a number of problems, for instance in random trapping problems. Consider a Poissonian point process with intensity $\beta > 0$ in $\mathbb{R}^d$. Around each point we put a ball of radius $a > 0$. These balls act as traps. Independently of this point process, we consider a standard Brownian motion. Define the trapping time τ as the first encounter of the Brownian with one of the traps. One is interested in $P\,(\tau > T)$, i.e. the (small) probability that no trapping occurs up to time T. P here refers to the joint distribution of the traps and the Brownian motion. To calculate it, we can integrate out the Poisson process first. Evidently, there is no trapping if no point of the point process falls into an a-neighborhood of the path of the Brownian, i.e. if the set W_T^a is trap free. The probability (under the point process) that this happens is $\exp\left[-\beta V_T^a\right]$. Therefore

$$P(\tau > T) = E\exp\left[-\beta V_T^a\right].$$

Here, on the right hand side, E refers to just taking Brownian expectation. In random media, such a situation is called *annealed.* This refers to integrating out the trap configuration (the "random environment") together with the "random walk", here the Brownian motion. In contrast to this, one can consider the *quenched* situation. Here one would keep the environment, i.e. the trap configuration fixed, and asks about $P(\tau > T)$ in the $T \to \infty$ limit, where P now refers only to the Brownian. This quantity now depends on the realizations of the traps, so one should write $P_\omega(\tau > T)$, ω referring to the trap configurations, and one would then like to have the limiting behavior of this as $T \to \infty$, for almost all trap configurations. I will here entirely focus on the annealed situation. A detailed study of the quenched situation is done in [73].

Regarding $E\exp\left[-\beta V_T^a\right]$, the classical result of Donsker-Varadhan states:

Theorem 2.49. *For any $\beta > 0$*

$$\lim_{T\to\infty} \frac{1}{T^{d/(d+2)}} \log E\left(\exp\left[-\beta V_T^a\right]\right) = \psi(\beta),$$

where

$$\psi(\beta) = (\omega_d\beta)^{2/(d+2)}\, \lambda_d^{d/(d+2)} \left(\frac{2}{d}\right)^{d/(d+2)} \frac{d+2}{2},$$

ω_d being the volume of a ball of radius one and λ_d is the ground state eigenvalue of the $\frac{1}{2}\Delta$ in the ball with radius one with Dirichlet boundary conditions.

There is a similar result for N_T in the random walk case (see [35]).

In order to understand the result and especially the somewhat strange power of T appearing in this large deviation result, one has first to look at the lower bound. One seeming very crude bound is obtained by confining the Brownian motion inside a ball $B_{r_T}(0)$ whose radius r_T has to be determined. For such path the volume of the sausage certainly is not larger than the volume of $B_{r_T+a}(0)$ which is $\omega_d(r_T+a)^d$. On the other hand, it is well known that

$$P\left(\beta_s \in B_{r_T},\, s \leq T\right) \geq C\exp\left[-\lambda_d \frac{T}{r_T^2}\right].$$

Therefore, we get for any choice of r_T :

$$E\left(\exp\left[-\beta V_T^a\right]\right) \geq C \exp\left[-\beta\omega_d (r_T + a)^d - \lambda_d \frac{T}{r_T^2}\right].$$

Optimizing over r_T one finds that the optimal radius is $r_T \sim \left(\frac{2\lambda_d T}{d\beta\omega_d}\right)^{1/(d+2)} \stackrel{\text{def}}{=} \rho(\beta)T^{1/(d+2)}$, which gives the lower bound in Theorem 2.49. The difficult part of the theorem is of course the upper bound. It might look somewhat surprising that the above crude argument for the lower bound gives the correct asymptotics, at least in leading order. In order to prove an upper bound one would like to argue roughly as follows:

$$\begin{aligned} E\left(\exp\left[-\beta V_T^a\right]\right) &= \sum_A P(W_T^a = A) \exp\left[-\beta|A|\right] \\ &\leq \sum_A P(W_T^a \subset A) \exp\left[-\beta\,|A|\right] \\ &\simeq \sum_A \exp\left[-\beta|A| - \lambda(A)T\right], \end{aligned}$$

where $\lambda(A)$ is the Dirichlet eigenvalue in A. Of course, the summation does not quite make sense, but it should naturally be understood to run over unions of blocks of side length εa, ε small, of a fixed grid.

The main problem is that the sum is running over too many sets. The relevant A's are roughly of diameter $T^{1/(d+2)}$ where both $|A|$ and $\lambda(A)T$ are typically of order $T^{d/(d+2)}$. Therefore, there are $\exp\left[CT^{d/(d+2)}\right]$ connected A's which are of the relevant size, so it is clear that one needs some coarse-graining in order to reduce the combinatorial complexity of the summation. It is natural that such a coarse-graining should be possible as the Brownian motion (or the random walk) is smearing out the empirical measure to some extent, so one can believe that one does not really have to sum over so many possibilities. This is also one of the basic ideas of the enlargement of obstacles technique by Sznitman (which works also in the quenched random trap situation not discussed here). I will give no details of these techniques, but will explain in the next section a new approach which has been developed in the "critical" case, recently. Anyway, if one is ready to believe that such a coarse-graining works, one gets

$$E\left(\exp\left[-\beta V_T^a\right]\right) \stackrel{\log}{\sim} \exp\left[-T^{d/(d+2)} \inf_A \left\{\beta|A| + \lambda(A)\right\}\right],$$

where $\lambda(A)$ is the Dirichlet ground state eigenvalue of $\Delta/2$ in A, and where $\stackrel{\log}{\sim}$ means that the quotient of the logarithms is going to 1. The variational problem above is a well known one in Mathematical Physics from the beginning of the century, which has been solved independently by Faber and Krahn, who proved that the unique minimizers are the balls. This is closely related to the classical isoperimetric problem, and can be reduced to it.

A problem in the spirit of the last section is to determine the behavior of the path measure

$$d\hat{P}_{T,\beta} = \frac{\exp\left[-\beta V_T^a\right] dP}{Z_T}$$

for large T. In the formulation as a trapping problem this would be the distribution of the Brownian motion, conditioned not to be trapped up to time T, in the annealed situation. From the Faber-Krahn Theorem it is natural to expect that the paths under this measure are concentrated on balls of radius about $\rho(\beta)T^{1/(d+2)}$. In particular this should mean that the path stays confined within a region of this order. However, even given the techniques to prove the Donsker-Varadhan results, this is far from being evident. The main delicacy is coming from the fact that one has to get some control over certain expectations beyond leading order asymptotics. In this respect the situation is quite similar to the one encountered in the last section, where the different possibilities on the fibers could not be distinguished by leading order asymptotics, too.

To see the difficulties in the present problem, consider the event that the Brownian path rushes off through a small tube (of radius 1, say) to a distance which is very large compared with $T^{1/(d+2)}$, to be specific, say to $\sqrt{T}$, and afterwards settles in an optimal ball such far out. This eccentricity gives a contribution of order $\sqrt{T}$ to the volume of the sausage, which may look large, but which is negligible when compared with the volume of the optimal ball, which is of order $T^{d/(d+2)}$. The probability for rushing (in time $\sqrt{T}$, say) through this narrow tube is for the standard Brownian of order $\exp\left[-\sqrt{T}\right]$, which may look small, but which is very large compared with the probability that the path does what we expect of it, namely to stay within the optimal ball, which is $\exp\left[-CT^{d/(d+2)}\right]$. The path could of course do many other things besides just this "tube eccentricity", and at the outset, it is not clear if one really should believe in this confinement (and had in fact been doubted by experts in the beginning).

The problem had first been addressed independently for $d = 2$ in two papers, the first one by Sznitman [72] and then in [11] for the random walk case. (The first versions of the two papers came out at about the same time.) The confinement has now been proved in a recent paper by Povel [62], which is based on the approach by Sznitman. The results for $d \geq 3$ are still not quite as precise as the one for $d = 2$.

Theorem 2.50. *There exists a function $\delta(T) \to 0$, as $T \to \infty$, such that*
a) *for $d = 2$*

$$\lim_{T\to\infty} \hat{P}_T\left(\exists x \in B_{\rho T^{1/(d+2)}}(0) : B_{\rho(1-\delta(T))T^{1/(d+2)}}(x) \subset W_T^a \subset B_{\rho(1+\delta(T))T^{1/(d+2)}}(x)\right) = 1$$

b) *(Povel [62]) for* $d \geq 3$:

$$\lim_{T\to\infty} \hat{P}_T \left(\exists x \in B_{\rho T^{1/(d+2)}}(0) : W_T^a \subset B_{\rho(1+\delta(T))T^{1/(d+2)}}(x)\right) = 1.$$

Sznitman's result contains also the limiting distribution of the centering of the optimal ball, which is not at 0, but which is distributed (after rescaling space with $\left(\rho T^{1/(d+2)}\right)^{-1}$) to the normalized ground state eigenfunction of $\Delta/2$ inside the unit ball.

There is no serious doubt that the full result, i.e. also a) and including the limiting distribution, is true in all dimensions, and could probably be proved by some additional efforts. The information on $\delta(T)$ is still very modest. The only information which is known is that one can take some decay of the form $T^{-\alpha}$, for *some* $\alpha > 0$. Bounds for α could be given, but they certainly are not optimal. It seems to be completely out of reach by present day's methods to get the precise behavior of the boundary, not even for $d = 2$. An interesting aspect, however, is the proof of such a droplet construction in sup-distance in any dimension.

It is fairly clear that a complete expansion of $E\left(\mathrm{e}^{-\beta V_T}\right)$ up to order $(1 + o(1))$ would be very helpful and desirable for the problem, but this seems to be completely out of reach, too. The methods in [13] do not apply, because V_T as a function of the empirical distribution has very bad continuity properties. The best results so far is the one obtained in [11] for the random walk where one takes

$$N_T \stackrel{\text{def}}{=} \#\left\{x \in \mathbb{Z}^d : X_s = x \,\text{for some}\, s \leq T\right\},$$

instead of V_T. The rough large deviation result (Theorem 2.49) is exactly the same as in the sausage case. The following sharpening of the statement in Theorem 2.49 is proved (in the random walk case) for all dimensions in [11], provided the variational problem has a rigidity property of the form of Theorem 2.51 below (which I hadn't known to be proved when writing the paper). The statement is that there exist c_1, c_2, ε (depending on d and β) such that

$$\begin{aligned}
&\exp\left[-\psi(\beta)T^{d/d+2)} - c_1(\beta)T^{(d-1)/(d+2)}\right] \\
&\leq E\left(\exp\left[-\beta N_T\right]\right) \\
&\leq \exp\left[-\psi(\beta)T^{d//d+2)} + c_2(\beta)T^{(d-\varepsilon)/(d+2)}\right].
\end{aligned} \tag{2.40}$$

ε can be estimated but presently, there is no hope getting the correct ε. There is a non-rigorous calculation in the physics literature [47], claiming that the correct correction is of the form of the lower bound:

$$\begin{aligned}
&E\left(\exp\left[-\beta N_T\right]\right) \\
&\qquad = \exp\left[-\psi(\beta)T^{d/d+2)} + c_1(\beta)T^{(d-1)/(d+2)} + o(T^{(d-1)/(d+2)})\right],
\end{aligned}$$

but this is based on some Gaussian Ansatz for the field of local times, and I do not know how reliable this prediction is. If correct, this would mean that the correction to the volume order large deviations is of surface order. This is a very interesting open problem.

One crucial ingredient in all the proofs of results like Theorem 2.50 is an analytic rigidity property of the variational problem, which in our case can be reduced to a rigidity property in the classical isoperimetric problem. This property states that if there is a (nice) subset A in $\mathbb{R}^d$ which has as volume that of the ball of radius one, and a surface which is slightly larger, then there exists a ball with radius one which is close in some sense to A. There is a substantial difference between $d = 2$ and $d \geq 3$. For $d = 2$ such a statement can easily been proved in Hausdorff-distance (with the help of the Bonnesen inequality), but in higher dimension, this evidently cannot be true. In fact, for $d \geq 3$, there are sets A with thin spines, these spines having essentially no volume and surface. It is therefore clear that such a rigidity can only be true in some L_1-sense. The following result has been proved by Hall [54].

Theorem 2.51. *Let ω_d be the volume of the ball with radius 1, and σ_d its surface. There exist $c(d)$, $\alpha(d) > 0$ such that for any Borel subset A of $\mathbb{R}^d$ with rectifiable boundary ∂A which satisfies $|A| = \omega_d$, there exists $x \in \mathbb{R}^d$ such that*

$$|A \,\Delta\, B_1(x)| \leq c(d)(|\partial A| - \sigma_d)^{\alpha(d)}.$$

It is possible to derive from this a corresponding rigidity result in L_1 for the variational problem appearing in the Donsker-Varadhan result

$$\psi(\beta) = \inf_{\|g\|_2=1} \left\{ \frac{1}{2} \int |\nabla g|^2 + \beta \int 1\{g^2 > 0\} dx \right\}.$$

This has been done for $d = 2$ in [11], but the proof there works in all dimensions, given the above theorem. The solutions of this variational problem are unique modulo shifts and given as the ground state eigenfunctions over the ball with optimal radius $\varrho(\beta)$, (i.e. just the usual Bessel function). This is the content of the celebrated Faber-Krahn Theorem. Let $\mathcal{F}$ be the set of squares of these optimal profiles. From Theorem 2.51 one can derive (this is not completely evident)

Proposition 2.52. *There exists $\delta > 0$, and $c(\beta) > 0$ such that*

$$\begin{aligned} \inf\{\frac{1}{2} \int | \nabla g|^2 + \beta \int 1_{\{g^2>0\}} dx : \int g^2 dx = 1, \inf_{f \in \mathcal{F}} \|f - g^2\|_1 \geq a\} \\ \geq \psi(\beta) + c(\beta) a^{\delta}. \end{aligned}$$

(see Lemma 3.1 in [11]).

This rigidity property implies in any dimension a corresponding probabilistic property in L_1. Due to bad continuity properties of the Lebesgue measure of the support, even this is not completely evident. However, the *main* difficulty is to improve this L_1-version to an L_∞-version.

I will give a short outline of this in the random walk case. To state the L_1-version, first a compactification is convenient, which is just the usual torus compactification. Fix some multiple of the optimal radius $\varrho(\beta)$, $R = 10\rho(\beta)$, say. Then we perform the usual periodization on a torus of side-length $RT^{1/(2+d)}$, and we scale everything down to a torus of finite side-length R, by replacing the random walk $X_t, t \geq 0$, on the torus by

$$\eta_t = T^{-1/(d+2)} X_{tT^{2/(2+d)}},$$

living on $\mathbb{L}_T^{(R)} = T^{-1/(2+d)}\{1, \ldots, RT^{1/(2+d)}\}^d$. This is now a process which is running on a torus of fixed size (but with grid which becomes finer and finer). Remark also that the total time for the rescaled process is

$$\tau \overset{\mathrm{def}}{=} T^{d/(d+2)}.$$

Next, we consider the "local times"

$$\ell_T^{(R)}(x) = \int_0^{T^{d/(2+d)}} 1_x(\eta_s)ds,$$

$x \in \mathbb{L}_T^{(R)}$. Remark that $\ell_T^{(R)}$ is normalized in the usual sense:

$$\int \ell_T^{(R)}(x)dx = 1,$$

if $\ell_T^{(R)}$ is extended to the continuous torus $[0, R)^d$ by putting it constant on the plaquette of side length $T^{-1/(2+d)}$. The probabilistic counterpart of the above Proposition 2.52 is the following result which has been proved in [11].

Proposition 2.53. *There exists* $\delta > 0$ *such that*

$$\lim_{T\to\infty} \hat{P}_{T,\beta}(\|\ell_T^{(R)} - \mathcal{F}\|_1 \geq T^{-\delta}) = 0.$$

There is of course no problem to define $\mathcal{F}$ on the torus, as the members of $\mathcal{F}$ (on $\mathbb{R}^d$) have as support the balls of radius $\varrho(\beta)$.

A consequence of this is that (on our torus) most of the mass of $\ell_T^{(R)}$ is concentrated inside a ball of the optimal radius. However, we are not really interested in the torus situation, and it is not clear that the above result should give us *anything* for the non-compact case. The crucial problem is to boost the result on the torus to a proof that there is *no* mass outside a ball of radius $\rho(\beta)$. Once one has proved this confinement property for the random walk on the torus, meaning in particular, that the confining ball most contain the starting point, it is evident, that one has also proved that the original problem (on $\mathbb{Z}^d$) has the corresponding confinement property, and therefore, Theorem 2.50 follows, at least the statement in part b).

The method how to achieve this had been different in Sznitman's (and now in Povel's) papers, and in [11]. Roughly speaking, in [11] it was done by "bare hand", whereas in [72] and [62] this came out from the enlargement of obstacles technique, together with some other non-trivial considerations.

I quickly sketch the main idea used in [11], which might be useful in other contexts, too. It is based on an iterative procedure. The above proposition implies that the total time spent outside a ball of optimal radius ρ is bounded by some $\tau^\eta = T^{\eta d/(d+2)}$, $\eta < 1$ in the time scale of the rescaled walk. We would like to exclude the possibility that there is *any* time spent outside. Now assume that the path really spends time $T^{\eta d/(d+2)}$ outside the ball, but that it is very pleasant and does this just at the end, and stays confined up to time $T^{\eta d/(d+2)} - T^{d/(d+2)}$. A moment's reflection shows if the path could only do such a thing, this could easily be excluded. We can separate what the path before and after $T^{\eta d/(d+2)} - T^{d/(d+2)}$ is doing (we used such arguments in the lest section). Arguing now anew with the shorter path of length $T^{\eta d/(d+2)} \ll T^{d/(d+2)}$, we can conclude that under the Gibbs measure, in reality, it would be essentially be confined to a ball of radius $\ll T^{1/(d+2)}$, doing some "really" nasty things just on still a smaller piece, hopefully being pleasant enough to do it at the end such that we can iterate the argument until we have shown that there can be no excursion at all.

The trouble is of course that a priori the path has no reason to be so pleasant to do the nasty things just in one piece. There could be many pieces, starting and entering from remote points on the surface of the droplet, and doing all kind of pranks. The argument to get this under control was roughly as follows. One introduces a (finite) number of radii slightly lager than the optimal $\varrho(\beta) = r_0 < r_1 < r_2 < \ldots r_m$. Actually the differences $r_i - r_{i-1}$ can be chosen to decay with T. Then one knows from Proposition 2.53 that the total time spent outside r_1 by the rescaled walk is at most $T^{d\eta/(d+2)}, \eta < 1$. This can then be boosted to prove that outside r_2 there is still less, namely $\leq T^{d\eta_2/(d+2)}, \eta_2 < \eta_1$. The reasoning roughly is that one can separate what is outside r_1 from what is inside and argue as before. One can proceed in this way and prove that outside r_m, there is nothing left. The proof in [11] of this inductive cutting of the excursions was quite involving and depended besides Theorem 2.51 on some previous knowledge which was very easy for $d = 2$, but which was not done in higher dimensions.

On the whole, the Sznitman argument was considerably more elegant, but it uses quite special properties. In particular the fact that the ground state eigenvalue is very strongly tied to probabilistic properties was used heavily. The argument sketched above is essentially just a complicated counting argument and might be of use in other problems.

2.6 Moderate deviation for the Wiener sausage

I discuss in this section a recent result obtained together with Michiel van den Berg and Frank den Hollander [6] on what one might call the "critically shrinking Wiener sausage". It will become clear later on in which sense the situation is "critical". The interest in this special case is coming from the fact that it is the border-line case where the Donsker-Varadhan behavior, which played a major rôle in the last section, starts to break down, and where the droplet we have discussed in the last section starts to "dissolve" in a way which will become clear.

2.6.1 Introduction and heuristics

In the classical Donsker-Varadhan result for the Wiener sausage discussed in the previous section, the main contribution to $E\left(\exp\left[-\beta V_T\right]\right)$ was coming from paths which stay inside a ball of radius $r_T = \rho(\beta)T^{1/(d+2)}$. The "strategy" the path has to follow is somehow trivial: The ball is filled just completely. Even if this is not fully proved in all dimensions, the fact that the trivial lower bound is correct in first order tells us that this is at least up to leading order the correct picture.

Consider first a *much* easier problem namely a Brownian motion, which is conditioned to stay inside a ball of radius T^γ. What is the effect on V_T of this conditioning? It is well known that under Wiener measure, V_T is typically of order $\kappa_a T$, for $d \geq 3$, where κ_a is the Newtonian capacity of a ball with radius a (see [67], [49], there is a logarithmic correction for $d = 2$). If the Brownian is confined in this ball, the volume can be at most of order $T^{d\gamma}$. Therefore, this confinement has trivially a substantial effect on the volume when $0 < \gamma < 1/d$, and it is not difficult to prove that a sausage of the Brownian which is confined to stay inside such a small ball is filling it completely, except near the boundary. Therefore, the volume is (up to smaller order corrections) just the volume of the ball. Let's look now at the opposite situation where $\gamma > 1/d$. In that case, of course, the volume of the ball is much larger than the expectation of the sausage it has when not confined, although for $d \geq 3$, and $\gamma < 1/2$, confining the Brownian to stay inside $B_{T^\gamma}(0)$ is still a large deviation. It is however not very difficult to see that the confinement in this case has no effect in leading order on the volume of the sausage. i.e. $E\left(V_T \,|\, W_T \subset B_{T^\gamma}(0)\right) = \kappa_a T + o(T)$. I don't know of a reference for this claim, and I don't want to prove it here, but the reader can easily convince himself of this fact. It is therefore clear that the critical confinement radius which should lead to a sizeable effect on the sausage is of order $T^{1/d}$. It is also not difficult to prove that the Brownian motion conditioned to stay inside a ball of radius $T^{1/d}$ $(d \geq 3)$ has an expectation of the sausage of order T but smaller than $\kappa_a T$: $E\left(V_T \,|\, W_T \subset B_{T^{1/d}}(0)\right) = aT + o(T)$ with $a < \kappa_a$.

Let's now go back to the (much more difficult problem) to discus $E\left(\exp\left[-\beta T^{-\alpha} V_T\right]\right)$. If we proceed with the lower bound explained in the last section, the optimal radius to choose will be the one where $T^{-\alpha}r_T^d$ is of order T/r_T^2, i.e. where r_T is of order $T^{(1+\alpha)/(2+d)}$. Now, this radius becomes of order $\geq T^{1/d}$ when $\alpha \geq 2/d$. It can therefore be expected that for $\alpha < 2/d$, the Donsker-Varadhan picture stays correct, and

$$\lim_{T\to\infty} T^{-(d-2\alpha)/(d+2)} \log E \exp\left[-\beta T^{-\alpha} V_T\right] = \psi(\beta).$$

This is indeed true, and has been proved independently in two papers ([10], [70]). The original Donsker-Varadhan approach however does not immediately extend to this situation and has to be refined. For $\alpha > 2/d$ this "Donsker-Varadhan ball picture" breaks down, which should be quite natural given the above discussion. In fact in leading order, the lower bound coming from the Jensen inequality is better than the one coming from the "ball strategy" and turns out to be sharp at least in leading order:

$$E \exp\left[-\beta T^{-\alpha} V_T\right] \overset{\log}{\sim} \exp\left[-\beta T^{-\alpha} E(V_T)\right] \sim \exp\left[-\beta \kappa_a T^{1-\alpha}\right].$$

The fact that Jensen is sharp can only mean that it is not "worth" for the Brownian to make any efforts and therefore that the corresponding path measure

$$d\hat{P}_T = \exp\left[-\beta T^{-\alpha} V_T\right] dP_T / Z_T$$

should just be close to ordinary Brownian. This has not been proved, and in view of the discussion given in Section 2.3, one would probably have to prove first that

$$E \exp\left[-\beta T^{-\alpha} V_T\right] \leq C \exp\left[-\beta \kappa_a T^{1-\alpha}\right],$$

which has not been done. Anyway, the most interesting case is certainly $\alpha = 2/d$, where we now have two lower bounds, one coming from Jensen, and the other one from the ball confinement strategy. On the background of the fact that a Brownian which is conditioned to stay in the ball of optimal radius does not fill the ball completely, one would certainly not expect the lower bound to be sharp (in leading order). Somewhat surprisingly, it turns out that the Jensen inequality is sharp for small β, but not for large, where something more interesting is happening, and where also the ball strategy is not the proper thing. This will become clear later.

It turns out that we better do not start with discussing $E \exp\left[-\beta T^{-2/d} V_T\right]$, but rather with a problem which looks equivalent, but isn't quite, namely with the probability that V_T is small in a range which would correspond to this critical case. It is natural to expect that the discussion of $E \exp\left[-\beta T^{-2/d} V_T\right]$ is tied to the question of discussing $P(V_T \leq bT)$, where $b < \kappa_a$. In fact, we can evaluate $E \exp\left[-\beta T^{-2/d} V_T\right]$ in leading order from the evaluation of $P(V_T \leq bT)$, but not vice versa.

Theorem 2.54. *Assume $d \geq 3$. Then for $b \in (0, \kappa_a)$*

$$\lim_{T\to\infty} T^{-(d-2)/d} \log P\left(V_T \leq bT\right) = -I(b)$$

where

$$I(b) = \inf\left\{\frac{1}{2}\left\|\nabla g\right\|_2^2 : g \in H_1(\mathbb{R}^d), \|g\|_2^2 = 1, \int \left(1 - \mathrm{e}^{-\kappa_a g(x)^2}\right) dx \leq b\right\} > 0. \tag{2.41}$$

Remark 2.55. a) There is also a version for $d = 2$. In that case, $E(V_T) \sim \kappa T/\log T$, where κ is the logarithmic capacity. The Theorem has then to be modified accordingly, i.e. one discusses $P\left(V_T \leq bT/\log T\right)$.

b) It is easy to evaluate $E \exp\left[-\beta T^{-2/d} V_T\right]$ using Theorem 2.54:

$$\lim_{T\to\infty} T^{-(d-2)/d} \log E \exp\left[-\beta T^{-2/d} V_T\right] = -J(\beta), \tag{2.42}$$

where J is the Legendre transform of I :

$$J(\beta) = \inf\left\{b\beta + I(b) : b \in (0, \kappa_a]\right\}, \tag{2.43}$$

but not the other way: I is not the Legendre transform of J. This is simply coming from the fact that I is not convex (whereas J is). This will become apparent below. It will also turn out that for small β, the infimum is attained at $b = \kappa_a$, so that for small β one has $J(\beta) = \kappa_a \beta$ ($I(\kappa_a)$ is of course 0), i.e. the Jensen inequality is sharp in leading order.

c) Presently, we are not able to discuss the path measures, for instance discuss the limiting behavior of the distribution of the end point β_T under $P(\cdot|V_T \leq bT)$ or under $d\hat{P}_T = \exp\left[-\beta T^{-2/d} V_T\right] dP/Z$. From the discussion in the last section it should be clear that the measures are living on scale $T^{1/d}$, i.e. one would expect that $T^{-1/d}\beta_T$ has under these measures a nontrivial limiting distribution. For $\hat{P}_T$ however, there should be a "collapse transition" from small to large β. In the region where $J(\beta) = \kappa_a \beta$, i.e. for small β, one would expect diffusive behavior, and only for β large, one would expect a subdiffusivity. However, most probably, there are further complications for $d \geq 5$, for reasons which will become apparent in the next section. Nothing on the path measures is proved, and it may be quite difficult.

d) The result can easily been extended to more general "sausages" where the ball with radius r is replaced by an arbitrary compact set C with positive capacity, i.e. where $W_T = \bigcup_{s\leq T}(\beta_s + C)$. Remark also that the rate function I does depend (via the capacity) on this compact set. This is not done in [6], but it follows by the same method.

I first will give an intuitive explanation why the above large deviation principle should hold and why the variational problem looks as it does. Afterwards, I will present in Subsection 2.6.2 the main analytical properties of the variational problem which are quite interesting and surprising. I will not give detailed proofs, but some explanations which hopefully will convince the reader, that the results have to be true. It is clear that the properties of the variational problem should be reflected also in properties of the path measure, but as remarked above, we don't know how to prove this. Especially, the somewhat strange behavior of the variational problem for $d \geq 5$ we will encounter must be reflected in an equally strange behavior of the path measure. I will then give a fairly detailed proof of the interesting probabilistic part of the Theorem 2.54, namely the upper bound in Section 2.6.3.

I start with giving a heuristic derivation why the rate function should have the above form.

From the discussion previous to the statement of the theorem, it should be apparent that the main contribution to the event $\{V_T \leq bT\}$ is coming from paths which are staying at distance of order $T^{1/d}$ from the origin. Furthermore, it should also be clear, that we no longer can expect that the "strategy" of the Brownian being as simple as to just fill a certain region completely, essentially without leaving holes. In contrast, we expect that there remains some porosity, and we have to control the degree of this porosity. The reason that we expect such a porosity is simply coming from the fact that a Brownian motion conditioned to stay inside a ball of radius $T^{1/d}$ exhibits such a porosity. This can easily be checked (but of course does not prove that such an effect is happening in our problem, too). This porosity is however felt only on a very microscopic scale: It turns out that the holes which are of relevance and are responsible for the porosity have size of order one. What we prove is essentially that the degree of the porosity is tied to empirical distribution at a macroscopic scale (i.e. $T^{1/d}$) deterministically, up to a superexponential estimate.

We first rescale the Brownian motion accordingly, by introducing $\widetilde{\beta}_t = T^{-1/d}\beta_{tT^{2/d}}$, $t \leq \tau \stackrel{\text{def}}{=} T^{(d-2)/d}$. As τ is the "correct time scale", we keep this notation in this way, and use τ always for this. Consider the empirical process

$$L_\tau \stackrel{\text{def}}{=} \frac{1}{\tau}\int_0^\tau \delta_{\widetilde{\beta}_s}\, ds.$$

By a (weak) LDP, we know that roughly speaking

$$P\left(L_\tau \sim f^2\right) \sim \exp\left[-\tau \frac{1}{2}\left\|\nabla f\right\|_2^2\right].$$

It is however not quite clear what L_τ has to do with the volume of the Wiener sausage. Remember that $\widetilde{\beta}_t$ is scaled down by a factor $T^{-1/d} = \tau^{-1/(d-2)}$ in space, and therefore

$$V_T = T\left|\operatorname{supp}\left(\chi_{B_{a\tau^{-1/(d-2)}}} * L_\tau\right)\right|,$$

where $B_r(x)$ is the ball with radius r and center x, $B_r = B_r(0)$, χ_A is the indicator function of the set A, and $(f * \mu)(x) = \int f(x-y)\mu(dy)$. There is evidently some trouble as $\mathcal{M}_1^+(\mathbb{R}^d) \ni \mu \to \left|\text{supp}\left(\chi_{B_{a\tau^{-1/(d-2)}}} * \mu\right)\right|$ depends certainly not continuously on μ, and furthermore depends on τ.

We call $\tau^{-1/(d-2)}$ the *microscopic* scale. Let's look at a small but macroscopic box, i.e. we consider a hypercube Q of side-length ε and center $x \in \mathbb{R}^d$: $Q = \prod_{i=1}^d [x_i - \varepsilon/2, x_i + \varepsilon/2)$. $L_\tau(Q)$ measures the relative amount of time, the rescaled Brownian $\widetilde{\beta}_t, t \leq \tau$, spends inside Q. Evidently, this total amount will usually be cut into many time pieces, the Brownian exiting and reentering the cube. We make a number of *very* simplifying (false) assumptions: First, we pretend that Q is not a cube, but a torus of the same size with periodic boundary conditions. Next we assume that these many pieces of the Brownian inside Q are just one piece of a Brownian on this torus running up to time $L_\tau(Q)\tau$. We will then make this assumption for a collection of Q's which cover the space and patch things together, but let's first discuss the problem how much of our Q, which is now a torus, is covered by the (shrinking) sausage. We might hope that the calculation of the expectation is sufficient, and this in fact will turn out to be correct. This may be somewhat surprising as, after all, we are after a large deviation phenomenon, and so we may expect that deviations from expectations will play a rôle. However, we will prove that the deviations of the volume of the *microscopic* sausage on small *macroscopic* boxes from its expectation can be estimated on a superexponential scale in τ if the boxes are small ("mesoscopic"). Therefore, we first calculate the expectation of the volume of our (critically shrinking) Wiener sausage (with radius $a\tau^{-1/(d-2)}$), where the Brownian is running on a torus of side-length ε, and the total time is $\lambda\varepsilon^d\tau$. Lets denote this volume by X. As we have made all kinds of (false) assumptions, we can as well add one more, namely to have the uniform distribution as the starting measure.

$$\begin{aligned} EX &= \int_Q dx\, P\left(\exists s \leq \lambda\varepsilon^d\tau : \beta_s \in B_{a\tau^{-1/(d-2)}}(x)\right) \\ &= |Q|\left(1 - P\left(\beta_s \notin B_{a\tau^{-1/(d-2)}}(x), \forall s \leq \lambda\varepsilon^d\tau\right)\right). \end{aligned}$$

We now chop the time interval $[0, \lambda\varepsilon^d\tau)$ into many pieces of large length K, which we assume not to grow with τ. The probability that the Brownian (with uniform starting distribution) hits $B_{a\tau^{-1/(d-2)}}(x)$ in the time slot $[0, K)$ is $\frac{K\kappa_a}{\varepsilon^d\tau} + o(\tau^{-1})$. If the Brownian does not hit the ball in the first interval, it gets a next change in the second. The conditioning on non-hitting in the first, does not much influence the distribution, as the ball which has to be hit is small anyway. Therefore, we get approximately the same chance for the second slot which is essentially independent of the first one, and so on. We therefore have

$$P\left(\beta_s \notin B_{a\tau^{-1/(d-2)}}(x), \forall s \leq \lambda\varepsilon^d\tau\right) \simeq \left(1 - \frac{K\kappa_a}{\varepsilon^d\tau}\right)^{\lambda\varepsilon^d\tau/K} \simeq \exp\left[-\kappa_a\lambda\right]$$

and therefore

$$EX \simeq \varepsilon^d (1 - \exp[-\lambda \kappa_a]).$$

We now chop $\mathbb{R}^d$ into cubes Q_i of the above size, and *assume* for the moment that $L_\tau(Q_i) \simeq \lambda_i \varepsilon^d \tau$. Then the sausage fills, up to superexponential estimates for the probability not doing so (if the reader believes in what was said above), the Q_i with a proportion $1 - \exp[-\lambda_i \kappa_a]$. Therefore, the total volume covered is $\sum_i \varepsilon^d (1 - \exp[-\lambda_i \kappa_a])$.

This does all the job on the microscopic scale, and the large deviation we are after is now only a large deviation on the macroscopic scale, i.e. a standard large deviation for L_τ which is governed by the classical Donsker-Varadhan LDP. We have to sum over all possibilities for choosing the λ_i but according to standard wisdom in large deviations, only the maximum counts, and we get

$$\begin{aligned} P(V_T \leq bT) &\simeq \max \left\{ P(L_\tau \sim f) : \int (1 - \exp[-\kappa_a f(x)]) dx \leq b \right\} \\ &\simeq \exp[-\tau I(b)], \end{aligned}$$

where

$$I(b) = \inf \left\{ \frac{1}{2} \|\nabla g\|_2^2 : \int \left(1 - \mathrm{e}^{-\kappa_a g^2(x)}\right) dx \leq b \right\}.$$

That's it, and there remains only to prove it.

I present the real core of the argument in subsection 2.6.3, taking however some (plausible and not too difficult) technical Lemmas for granted. Before starting with it, I want to give some information about the variational problem, which had been quite surprising (at least to us).

2.6.2 Analytical properties of the variational problem

I am discussing here the main analytic features of the variational problem (2.41). I will not give detailed proofs, as they partially are quite lengthy, but I will try to explain the main properties. There does not seem to be an explicit solution. It is not too difficult to prove (using standard techniques) that all maximizers of the variational problem are radially symmetric. In principle, one can then discuss the one-dimensional Euler equation, which is just a nonlinear second order differential equation, but this seems not to be of much help. For instance, we have been unable to prove that there is a unique maximizer (modulo shifts), and the problem does not appear to belong to a class which has been treated in the literature.

The behavior of $I(b)$ for $b \sim 0$ is easy and offers no surprise: The variational problem goes over (after a rescaling) into the variational problem for the classical Donsker-Varadhan situation. It is fairly evident what the best way is in which a normed L_2-function can achieve a small value of $\int \left(1 - \mathrm{e}^{-\kappa_a g(x)^2}\right) dx$, best in the sense of having small value of $\|\nabla g\|_2^2$: g just has to be essentially 0 outside some small ball. Inside the ball, g then is quite large, because of the restriction $\int g^2(x)dx = 1$. Therefore $1 - e^{\kappa_a g^2(x)}$ is there essentially 1 inside the ball. This means that for small b we have

$$I(b) \sim \inf\{\frac{1}{2}\| \bigtriangledown g\|_2^2 : \|g\|_2 = 1,\ |\operatorname{supp}(g)| \le b\}.$$

After rescaling, this leads to

Proposition 2.56. *For $b \to 0$*

$$I(b) \sim \frac{1}{2}\lambda_d(\omega_d b)^{-2/d}.$$

Much more interesting is the behavior for $b \sim \kappa_a$. We naturally expect that the relevant functions for the variational problem become flat as $b \uparrow \kappa$. Following this idea, one expects that we just may expand the exponential:

$$1 - \exp(-\kappa_a g^2) \simeq \kappa_a g^2 - \frac{1}{2}\kappa_a^2 g^4,$$

and replace the restriction by the corresponding restriction on the expanded expression. Implementing the above, we get

$$\int (1 - e^{-\kappa_a g^2})dx \simeq \kappa_a - \frac{\kappa_a^2}{2}\int g^4(x)dx.$$

This means that for $b < \kappa_a, b \sim \kappa_a$, we should have

$$I(b) \approx \inf\left\{\frac{1}{2}\| \bigtriangledown g\|_2^2 : \int g^2(x)dx = 1,\ \frac{\kappa_a^2}{2}\int g^4(x)dx = \kappa_a - b\right\}. \tag{2.44}$$

The trouble is that the r.h.s. is 0 for $d \ge 5$. This is well known, but I give the proof as it indicates how things should run for $d \ge 5$. The claim simply is that for any $a > 0$, and $d \ge 5$

$$\inf\left\{\frac{1}{2}\| \bigtriangledown g\|_2^2 : \int g^2(x)dx = 1,\ \frac{\kappa_a^2}{2}\int g^4(x)dx = a\right\} = 0. \tag{2.45}$$

Here is the sequence, which does the job: We choose a ball with radius $1/n$, and over this ball a circular cone of height a_n. This is g_n inside the ball. We will describe g_n outside in a moment. We choose a_n such that $\int_{B_{1/n}} g_n^4(x)dx \approx a$, i.e. $a_n \approx n^{d/4}$. At the boundary, g_n is not quite 0, but this will have no effect on the L_4-norm. The contribution to the L_2-norm from inside the ball is then negligible, and we choose g_n outside very flat, producing the necessary L_2-norm. It is clear that we can do that in such a way that this contributes nothing to the L_4-norm, and also nothing to $\| \bigtriangledown g\|_2^2$ (asymptotically). In this way we take care of $\|g_n\|_2^2$, and $\| \bigtriangledown g_n\|_2^2$ is now determined from what is happening inside the ball:

$$\int_{B_{1/n}} | \bigtriangledown g_n(x)|^2 dx \approx n^{-d}(na_n)^2 \approx n^{-d+2}n^{d/2}.$$

which goes to 0 for $d \geq 5$, proving the above claim. This reveals that our approach of expanding the exponential for $d \geq 5$ is a failure, and we will come back to this in a moment. For $d \leq 4$, this is however the correct procedure, and one can prove the following result:

Proposition 2.57. *Assume $d \leq 4$. Then as $b \uparrow \kappa_a$*

$$I(b) \sim 2^{-\frac{d-2}{2}} \kappa_a^{-4/d} (\kappa_a - b)^{2/d} \mu_d,$$

where

a) *for* $d \leq 3$ $\quad \mu_d = \inf\{\|\nabla g\|_2^2 : g \in H^1\left(\mathbb{R}^d\right), \|g\|_2 = 1, \|g\|_4 = 1\} > 0$
b) *for* $d = 4$ $\quad \mu_d = \inf\{\|\nabla g\|_2^2 : g \in D^1\left(\mathbb{R}^4\right), \|g\|_4 = 1\}.$

(For background material about the spaces $H^1\left(\mathbb{R}^d\right)$ and $D^1\left(\mathbb{R}^d\right)$, see [51])

A consequence of this proposition is that for $d = 3, 4,$ I is concave close to κ_a (of course, the above result does not quite prove this), and has infinite tangent at κ_a.

We come now to the case $d \geq 5$. The argument above leading to the conclusion (2.45) does of course not prove that $I(b) = 0$, simply because the functions in the sequence we have chosen had a high peak inside a small ball, and this peak was important for the result. Remark furthermore, that the whole L_2-norm was "leaking" to ∞ as $n \to \infty$. For the peak inside the ball, the expansion is evidently not the right thing to do, and therefore, (2.45) does not give any immediate indication what the behavior of $I(b)$ should be. In fact $I(b) > 0$ for all $b \in (0, \kappa_a)$. There is however one feature of the above considerations which are important for the behavior of $I(b)$, $b \sim \kappa_a$, namely the possibility that L_2 is leaking to infinity (which happens for the sequence g_n). To catch this, we apply a trick. For $\int g^2 dx = 1$, we have $\int (1 - e^{-\kappa_a g^2}) dx = u$ if and only if

$$\int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2}) dx = \kappa_a - u.$$

The integrand has the advantage that it decays with g^4 if g is small. If therefore L_2-mass of g is wandering to infinity, this is not visible in the integrand, meaning that the integrand would behave continuously, although the L_2-norm would jump. We can therefore try to look at the variational problem forgetting for the moment the $\|g\|_2 = 1$ condition, i.e. look at

$$\varrho(\varepsilon) = \inf\{\frac{1}{2}\|\nabla g\|_2^2 : \int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2})dx = \varepsilon\}.$$

This problem is "well posed": one can prove that minimizers exist, and the infimum is > 0. In fact, the ε dependence is trivial, and can be obtained by a rescaling

$$\varrho(\varepsilon) = \varepsilon^{(d-2)/d}\varrho(1),$$

but it is crucial for this that we have left out the condition $\|g\|_2 = 1$. The above equation simply follows from the following observation: If g satisfies $\int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2})dx = 1$ then $g_\varepsilon(x) = g(\varepsilon^{-1/d}x)$ satisfies

$$\int \left(\kappa_a g_\varepsilon(x)^2 - 1 + e^{-\kappa_a g_\varepsilon(x)^2}\right) dx = \varepsilon,$$

and

$$\|\nabla g_\varepsilon\|_2^2 = \varepsilon^{\frac{d-2}{d}} \|\nabla g\|_2^2.$$

Unfortunately, we have not been able to prove that the variational problem for $\varrho(1)$ has a unique minimizer (modulo shifts), and we cannot exclude that there are several minimizers with different L_2-norm, although this does not look very plausible. One can however prove that there are minimizers, which are positive everywhere, and any minimizer has to be rotational symmetric. Let us *pretend* that there is (modulo shifts) just one or at least that all have the same L_2-norm. If this is not the case, the statement needs some messy but not very important modifications, and the outcome is essentially the same. Let therefore ψ_1 be the minimizer for $\varrho(1)$ (symmetric around 0, say). If we scale ψ_1 to serve for $\varrho(\varepsilon)$, i.e. take $\psi_\varepsilon(x) = \psi_1(\varepsilon^{-1/d}x)$, then

$$\|\psi_\varepsilon\|_2^2 = \varepsilon \|\psi_1\|_2^2 .$$

Now, our real problem is to determine

$$I(b) = \inf\{\frac{1}{2}\|\nabla g\|^2 : \|g\|_2 = 1, \int \left(\kappa_a g^2 - 1 + e^{-\kappa_a g^2}\right) dx = \kappa_a - b\}, \quad (2.46)$$

and it looks like that with have not gained very much to calculate $\varrho(\kappa_a - b)$ because $\psi_{\kappa_a - b}$ would only be the relevant minimizer if $(\kappa_a - b)\|\psi_1\|_2^2 = 1$. However, it turns out that if $(\kappa_a - b)\|\psi_1\|_2^2 < 1$, one in fact has

$$I(b) = \varrho(\kappa_a - b) = (\kappa_a - b)^{(d-2)/d}\varrho(1).$$

The point is that the variational problem (2.46) does in that case not have a minimizer, because L_2-mass is leaking out to infinity. Therefore, the relevant variational problem is simply the one without the L_2-restriction. This leads to the following conclusion (which is correct regardless of the uniqueness question).

Proposition 2.58. *Assume $d \geq 5$. Then there exists $b_0(d) \in (0, \kappa_a)$ such that for $b \in [b_0(d), \kappa_a]$ one has*

$$I(b) = (\kappa_a - b)^{(d-2)/d} \varrho(1).$$

In the case where $(\kappa_a - b)\|\psi_1\|_2^2 > 1$, which is true for b small, $I(b)$ has nothing to do with ϱ. The L_2-restriction then "deforms" ψ in an essential way. We also know that in this case the variational problem for $I(b)$ has solutions which have L_2-norm 1. For more details and proofs, see [6].

The previous claim that $J(\beta) = \kappa_a \beta$ (J from (2.43)) for small β now follows easily. From the fact that $I(\beta)$ has tangent ∞ at $\beta = \kappa_a$ implies that the infimum in (2.43) is attained in $\beta = \kappa_a$.

It is interesting to speculate what the behavior of the variational problem implies for the path measure. It should be evident that for $d = 3, 4$ and for $d \geq 5$ and b small, the paths under $P(\cdot | V_T \leq bT)$ are living on scale $T^{1/d}$, meaning for instance that

$$\sup_T E(T^{-1/d} |\beta_T| \; |V_T \leq bT) < \infty.$$

On the other hand, when $d \geq 5$ and b is close to κ_a, probably the behavior is different. The fact that the variational problem looses mass to infinity can only mean that the path stays "confined on scale $T^{1/d}$" only on part of its life time. For instance, one can imagine that the path first feels the confinement on a fixed proportion of T, and afterwards floats diffusively, but one could also imagine that a more complicated behavior emerges. All this would probably be very difficult to prove.

2.6.3 Proof of the upper bound in Theorem 2.54

I prove here the upper bound, except that I leave some technical lemmas unproved, but I will give some explanations for them.

It is convenient to use the usual trivial compactification procedure winding the Brownian motion on a torus. This we do however after having done the rescaling leading to $\widetilde{\beta}_s = T^{-1/d} \beta_{sT^{2/d}}$, $s \leq \tau = T^{(d-2)/d}$. We get $V_T^a = T V_\tau^{a\tau^{-1/(d-2)}}$. We wind the Brownian motion $\left\{\widetilde{\beta}_s\right\}_{s \leq \tau}$ on a torus Λ_N of fixed size with side length N. By an abuse of notation, we write $V_\tau^N \stackrel{\text{def}}{=} V_\tau^{a\tau^{-1/(d-2)}}$, but we also often drop the index N. Evidently, we have

$$P(V_T^a \leq bT) \leq P(V_\tau^N \leq b).$$

To get an upper bound of the left-hand side, we therefore have to bound the right-hand side. The main result to get that is:

Proposition 2.59. *V_τ^N satisfies a τ-large deviation principle with rate function*

$$I_N(a) = \inf\left\{\frac{1}{2}\int_{\Lambda_N} |\nabla g(x)|^2\, dx : g \in H_1(\Lambda_N),\right.$$
$$\left.\times \int_{\Lambda_N} \left[1 - \exp\left(-\kappa_a g^2(x)\right)\right] dx = a\right\}$$

where $H_1(\Lambda_N)$ is the usual Sobolev space of once weakly differentiable functions with derivative in $L_2(\Lambda_N)$.

The upper bound in our main Theorem 2.54 follows easily from this proposition. The only thing which remains is

Lemma 2.60. $\lim_{N\to\infty} I_N(a) = I(a)$ *for all* a.

I will not prove this lemma, which is not difficult.

A slight extension of the above proposition also leads to the lower bound in Theorem 2.54. I sketch the argument: Fixing a (large) number R, we have (on $\mathbb{R}^d$)

$$P\left(V_T \le bT\right) \ge P\left(V_T \le bT, \sup_{t\le T} |\beta_t| \le RT^{1/d}\right).$$

For the rescaled Brownian, the second event on the right-hand side is $\sup_{t\le\tau}\left|\widetilde{\beta}_t\right| \le R$. If we choose $N > R$, then it doesn't play any rôle whether the probability above is calculated for the torus Brownian motion or for the unconfined one. A slight extension of the proposition above gives for V_τ^N, conditioned on the event $\left\{\sup_{t\le\tau}\left|\widetilde{\beta}_t\right| \le R\right\}$ a τ-LDP with rate function

$$I_R\left(a\right) \overset{\mathrm{def}}{=} \inf\left\{\frac{1}{2}\int_{\Lambda_N} |\nabla g(x)|^2\, dx : \operatorname{supp}\left(g\right)\right.$$
$$\left.\subset \left[-\frac{R}{2}, \frac{R}{2}\right]^d, \int \left[1 - \mathrm{e}^{-\kappa_a g^2(x)}\right] dx = a\right\}.$$

Then

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T^{-(d-2)/d}} \log P\left(V_T \le bT\right) \\
&\ge \liminf_{T\to\infty} \frac{1}{T^{-(d-2)/d}} \log P\left(V_T \le bT \,\middle|\, \sup_{t\le T} |\beta_t| \le RT^{1/d}\right) \\
&+ \liminf_{T\to\infty} \frac{1}{T^{-(d-2)/d}} \log P\left(\sup_{t\le T} |\beta_t| \le RT^{1/d}\right) \\
&= -I_R\left(b\right) + \liminf_{T\to\infty} \frac{1}{T^{-(d-2)/d}} \log P\left(\sup_{t\le T} |\beta_t| \le RT^{1/d}\right).
\end{aligned}$$

The lower bound in Theorem 2.54 then follows by letting $R \to \infty$. I don't give the details which are not very interesting.

I will give the proof of the Proposition 2.59 in some details. From its form, it is clear that we should get it by a kind of contraction principle. It seems however impossible to get that directly, and we use an approximation procedure. For the rest of this chapter, the torus $\widetilde{\Lambda}_N$ is fixed. We usually drop N in the notation. We also drop the tilde in $\widetilde{\beta}_s$, and just write β_s for this rescaled Brownian motion. Time is always running up to τ.

Here is an outline of the procedure:

A) We first approximate $V_\tau (= V_\tau^N)$ by its conditional expectation

$$E_\varepsilon(V_\tau) = E(V_\tau | \{\beta_{i\varepsilon}\}_{0 \le i \le \tau/\varepsilon}),$$

where ε is a parameter > 0. We prove that the difference between V_τ and $E_\varepsilon(V_\tau)$ is negligible in the $\varepsilon \to 0$ limit. This is done by an application of a concentration inequality of Talagrand.

B) We represent $E_\varepsilon(V_\tau)$ as a functional of the empirical distribution

$$L_{\varepsilon,\tau} = \frac{\varepsilon}{\tau} \sum_{i=1}^{\tau/\varepsilon} \delta_{(\beta_{\varepsilon(i-1)}, \beta_{\varepsilon i})}.$$

According to one of the very basic large deviation results of Donsker and Varadhan, $L_{\varepsilon,\tau}$ satisfies for fixed ε a strong LDP (on the torus). We still will need some further approximations to get the dependence of $E_\varepsilon(V_\tau)$ on $L_{\varepsilon,\tau}$ in a suitable form, but essentially based just on this basic LDP for $L_{\varepsilon,\tau}$, we get via a contraction principle a LDP for $E_\varepsilon(V_\tau)$.

C) We finally have to perform the $\varepsilon \to 0$ limit. We now already know that V_τ is approximated by $E_\varepsilon(V_\tau)$. It therefore will suffice to have an appropriate transition for the variational formula.

We write $\mathbb{X}_{\tau,\varepsilon} = \{\beta_{i\varepsilon}\}_{1 \le i \le \tau/\varepsilon}$. (For notational convenience, we always assume that τ/ε is an integer). We denote by P_ε and E_ε the conditional probability and expectation with respect to $\mathbb{X}_{\tau,\varepsilon}$. The first main step (A) is to prove that V_τ is well approximated by $E_\varepsilon(V_\tau)$ in the following sense:

Proposition 2.61. *For all $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log P(|V_\tau - E_\varepsilon(V_\tau)| \ge \delta) = -\infty.$$

Proof. The proof is based on Talagrand's concentration inequalities. We denote by $m_{\tau,\varepsilon}$ the median of the distribution of V_τ under the conditional law P_ε. Furthermore, let W_i, $1 \le i \le \tau/\varepsilon$, be defined by

$$W_i = \bigcup_{s \in [(i-1)\varepsilon, i\varepsilon]} B_{a\tau^{-1/(d-2)}}(\beta_s). \tag{2.47}$$

Evidently, the W_i are, conditionally on $\mathbb{X}_{\tau,\varepsilon}$, are independent random closed subsets of Λ_N, and we have

$$V_\tau = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i \right|.$$

Let S be the set of closed subsets of Λ_N. The mapping $d : S \times S \to [0,\infty)$, $d(A,B) = |A \Delta B|$, defines a pseudometric on S. We equip S with the Borel field $\mathfrak{S}$ generated by this pseudometric. P_ε then defines a product measure on $(S,\mathfrak{S})^{\tau/\varepsilon}$, which, by an abuse of notation, we denote by P_ε, too. We apply one of Talagrand's concentration inequality to the function $V : S^{\tau/\varepsilon} \to [0,\infty)$, defined by

$$V(C) = \left| \bigcup_{i=1}^{\tau/\varepsilon} C_i \right|, \, C = \{C_i\}.$$

Evidently, V is Lipshitz in the sense that

$$|V(C) - V(C')| \leq \sum_{i=1}^{\tau/\varepsilon} |C_i \Delta C_i'| \, .$$

Let

$$A = \left\{ C \in S^{\tau/\varepsilon} : V(C) \leq m_{\tau/\varepsilon} \right\}.$$

The distribution of V under P_ε has no atoms. Therefore, we have $P_\varepsilon(A) = \frac{1}{2}$. From Theorem 2.4.1 of [74], we have

$$E_\varepsilon(\exp\left[\lambda f(A, \{W_i\})\right]) \leq 2 \prod_{i=1}^{\tau/\varepsilon} E_\varepsilon(\cosh(\lambda \left|W_i \Delta W_i'\right|)),$$

where $f(A,\{C_i\}) = \inf_{\{D_i\}\in A} \sum_i d(C_i, D_i)$ and $\{W_i'\}$ is an independent copy of $\{W_i\}$. From the Markov inequality, we therefore get

$$\begin{aligned} P_\varepsilon(f(A,\{W_i\}) \geq \delta) &\leq 2 \inf_{\lambda>0} e^{-\lambda\delta} \prod_{i=1}^{\tau/\varepsilon} E_\varepsilon(\cosh(\lambda \left|W_i \Delta W_i'\right|)) \qquad (2.48) \\ &= \Phi_{\tau,\varepsilon}(\delta), \text{say.} \end{aligned}$$

Arguing similarly with $A' = \left\{ C \in S^{\tau/\varepsilon} : V(C) \geq m_{\tau/\varepsilon} \right\}$, we get

$$P_\varepsilon(|V_\tau - m_{\tau,\varepsilon}| \geq \delta) \leq 2\Phi_{\tau,\varepsilon}(\delta).$$

Remark now that $|V_\tau|$ is bounded by $|\Lambda_N|$. Therefore

$$|E_\varepsilon(V_\tau) - m_{\tau,\varepsilon}| \leq \frac{\delta}{3} + 2\left|\Lambda_N\right| P_\varepsilon \left(|V_\tau - m_{\tau,\varepsilon}| \geq \frac{\delta}{3} \right).$$

Using this, we have

$$P_\varepsilon\left(|V_\tau - E_\varepsilon(V_\tau)| \geq \delta\right) \leq 2\Phi_{\tau,\varepsilon}\left(\frac{\delta}{3}\right) + I\left[P_\varepsilon\left(|V_\tau - m_{\tau,\varepsilon}| \geq \frac{\delta}{3}\right) \geq \frac{\delta}{6\,|\Lambda_N|}\right]$$
$$\leq 2\Phi_{\tau,\varepsilon}\left(\frac{\delta}{3}\right) + I\left[2\Phi_{\tau,\varepsilon}\left(\frac{\delta}{3}\right) \geq \frac{\delta}{6\,|\Lambda_N|}\right],$$

where $I[.]$, denotes the indicator function of an event. Using this inequality, we get

$$P(|V_\tau - E_\varepsilon(V_\tau)| \geq \delta) \leq 2\left(1 + \frac{6\,|\Lambda_N|}{\delta}\right) E\left(\Phi_{\tau,\varepsilon}\left(\frac{\delta}{3}\right)\right).$$

In order to prove the Proposition, it therefore suffices to show that

$$\lim_{\varepsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log E\left(\Phi_{\tau,\varepsilon}(\delta)\right) = -\infty \tag{2.49}$$

holds for all $\delta > 0$. We actually prove more, namely

$$\lim_{\varepsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log\|\Phi_{\tau,\varepsilon}(\delta)\|_\infty = -\infty. \tag{2.50}$$

To estimate $\Phi_{\tau,\varepsilon}(\delta)$ we will take $\lambda = \alpha\varepsilon^{-1}\tau$ with $0 < \alpha \leq 1$ in $E_\varepsilon(\cosh(\lambda\,|W_i \Delta W_i'|))$. Remark that $\cosh(\alpha b) \leq 1 + \alpha^2\exp(b)$, if $0 < \alpha \leq 1$ and $b > 0$. If $x \in \Lambda_N$, we write $E_{x,\varepsilon}$ for the expectation under a Brownian bridge on the Λ_N-torus, i.e. a Brownian motion $(\beta_s)_{0\leq s\leq\varepsilon}$ starting at 0 and conditioned to be at x at time ε. It is evident that the volume for the sausage of such a Brownian bridge on the torus is stochastically smaller than the corresponding sausage of a Brownian bridge on $\mathbb{R}^d$. We then have

$$E_\varepsilon(\cosh(\alpha(\tau/\varepsilon)\,|W_i\Delta W_i'|))$$
$$\leq 1 + \alpha^2\left(E_{\beta_{\varepsilon i}-\beta_{\varepsilon(i-1)},\varepsilon}(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right)^2,$$

where $W^a(t) = \bigcup_{s\leq t} B_a(\beta_s)$. As remarked above, we can replace the right hand side in the above inequality by the corresponding quantity for the standard Brownian motion, which has the advantage that we now can use the standard rescaling properties. Using these, we get

$$E_{x,\varepsilon}\left(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right)$$
$$\leq E^\infty_{\tau^{1/(d-2)}x,\varepsilon\tau^{2/(d-2)}}\left(\exp\left[\varepsilon^{-1}\tau^{-2/(d-2)}\left|W^a(\varepsilon\tau^{2/(d-2)})\right|\right]\right),$$

where P^∞, E^∞ refer to the Brownian on $\mathbb{R}^d$. According to the Lemma 2.62 below, we see that there is a $\tau_o(\varepsilon, N)$ such that for all $\tau \geq \tau_o(\varepsilon, N)$, all N , and all $x \in \Lambda_N$ we have

$$E^\infty_{x,\varepsilon}\left(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right) \leq C.$$

We therefore get

$$\prod_{i=1}^{\tau/\varepsilon} E_\varepsilon(\cosh(\lambda\,|W_i \varDelta W_i'|)) \le \prod_{i=1}^{\tau/\varepsilon}(1+\alpha^2 C^2) \le \exp\left((\tau/\varepsilon)\alpha^2 C^2\right).$$

Implementing it into (2.48), we get

$$\varPhi_{\tau,\varepsilon}(\delta) \le 2\exp\left[-\delta\alpha\frac{\varepsilon}{\tau} + C^2\alpha^2\frac{\varepsilon}{\tau}\right]$$

and choosing now α small enough (2.50) follows, and therefore the Proposition 2.61 is proved.

Lemma 2.62. *There exists a constant C with*

$$\sup_{t\ge 1, |x|\le t} E^{\infty}_{x,t}\left(\exp\left[\frac{1}{t}\,|W^a(t)|\right]\right) \le C.$$

I will not give a proof of this. For the unconditioned Brownian motion, this follows from estimates in [7]. The lemma states that the situation does not change much if we condition the Brownian to end in a point which is away from the starting point at maximum t. Although this is a large deviation for the Brownian, it is evident that this increases the sausage at maximum to something of order t, and so the statement of the Lemma looks plausible. It is not difficult to prove if by chopping time into small pieces.

We have finished the first part (A) of the proof, and we come to (B). During the proof of this part, we keep the parameter ε completely fixed.

We denote by p_s the transition densities for the Brownian motion (on the torus $\varLambda_N$, but as usual, we drop the N in the notation). For $y, z \in \varLambda_N$ we define

$$q^\varepsilon_b(y,z) = P(\exists s \le \varepsilon \text{ with } \beta_s \in B_b(0) | \beta_0 = y, \beta_\varepsilon = z),$$

and by an abuse of notation $q^\varepsilon_\tau(y,z) = q^\varepsilon_{a\tau^{-1/(d-2)}}(y,z)$ where a is the radius of the original sausage. We also set for $y, z \neq 0$

$$\varphi_\varepsilon(y,z) = \frac{\int_0^\varepsilon p_s(y)p_{\varepsilon-s}(z)\,ds}{p_\varepsilon(z-y)}.$$

It is evident (see below) that $E_\varepsilon(V_\tau)$ can be expressed with the help $q^\varepsilon_\tau(y,z)$ and the empirical measure $L_{\varepsilon,\tau}$, and we therefore easily get a LDP, except for the problem that $q^\varepsilon_\tau(y,z)$ still depends on τ. We don't like this τ-dependence. The basis for being able to remove it is the following technical result.

Lemma 2.63. a) *Let $b < b_1 < N/4$. Then*

$$\sup_{x,y\notin B_{b_1}} q^\varepsilon_b(x,y) \le C\left(\frac{b}{b_1}\right)^{d-2}$$

b) *For any* $\varepsilon, b > 0$ *we have*

$$\lim_{\tau\to\infty} \sup_{y,z\notin B_b(0)} \left|\tau q_\tau^\varepsilon(y,z) - \kappa_a \varphi_\varepsilon(y,z)\right| = 0,$$

where κ_a *is the Newtonian capacity of the ball with radius* a.

a) is rather evident and easy to prove. Remember that ε is fixed. The claim is that if the starting and the end point of the bridge are sufficiently for away from the ball to be hit, then there is only a small chance for this hitting. The exact form of the estimates comes easily from standard estimates of hitting probabilities.

b) is more delicate. From scaling, one sees that $q_\tau^\varepsilon(y,z)$ is in fact of order τ. The bridge has a chance to hit the small ball only if it already gets close to it. $\varphi_\varepsilon(y,z)$ measure the expectation of the total time, the bridge spends in the vicinity of the ball. This quantity has to be multiplied with the capacity of the ball, which is κ_a/τ. For details, see [6].

We now perform the approximation of $E_\varepsilon(V_\tau)$. We first approximate V_τ by cutting out small holes around the points $\beta_{i\varepsilon}$: Fix $b > 0$ and define

$$W_i^b = W_i \backslash \left(B_b(\beta_{(i-1)\varepsilon}) \cup B_b(\beta_{i\varepsilon})\right),$$

and set

$$V_\tau^K = \left|\bigcup_{i=1}^{\tau/\varepsilon} W_i^{K\tau^{-1/(d-2)}}\right|.$$

Evidently, we have cut out at maximum τ/ε times the volume of a ball of radius $K\tau^{-1/(d-2)}$. Therefore

$$\left|V_\tau - V_\tau^K\right| \le c\varepsilon^{-1} K^d \tau^{-2/(d-2)}, \tag{2.51}$$

and therefore the difference is negligible for our purpose. The cutting is convenient, because we can invoke then the Lemma 2.63 which helps to expand $\log(1-q)$ linearly in q just by $-q$.

$$\begin{aligned} E_\varepsilon(V_\tau^K) &= \int_{\Lambda_N} dx \left(1 - P_\varepsilon(x \notin \bigcup_{i=1}^{\tau/\varepsilon} W_i^{K\tau^{-1/(d-2)}})\right) \\ &= \int_{\Lambda_N} dx \left(1 - \prod_{i=1}^{\tau/\varepsilon} \left[1 - P_\varepsilon(x \in W_i^{K\tau^{-1/(d-2)}})\right]\right) \qquad (2.52) \\ &= \int_{\Lambda_N} dx \left(1 - \exp\left[\frac{\tau}{\varepsilon}\int \log\left(1 - q_\tau^{\varepsilon, K\tau^{-1/(d-2)}}(z-x, y-x)\right) L_{\varepsilon,\tau}(dy,dz)\right]\right), \end{aligned}$$

where $q_\tau^{\varepsilon,b}(z,y) = q_\tau^\varepsilon(z,y)$ if $z, y \notin B_b(0)$ and 0 otherwise. We are therefore naturally led to the investigation of mappings $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \to [0,\infty)$

$$\Phi_{\tau,\beta,b}(\mu) \stackrel{\text{def}}{=} \int_{\Lambda_N} dx \left(1 - \exp\left[-\beta\tau \int q_\tau^{\varepsilon,b}(z-x, y-x)\mu(dy,dz)\right]\right).$$

Then, we get the sandwiching

$$\Phi_{\tau,(1+\delta_K)/\varepsilon,K\tau^{-1/(d-2)}}(L_{\varepsilon,\tau}) \leq E_\varepsilon(V_K) \leq \Phi_{\tau,1/\varepsilon,K\tau^{-1/(d-2)}}(L_{\varepsilon,\tau}),$$

with $\delta_K \to 0$ for $K \to \infty$. This follows from Lemma 2.63, part a). With the same lemma, we also see that we can replace $K\tau^{-1/(d-2)}$ with a fixed (small) value b :

$$\left\| E_\varepsilon(V_K) - \Phi_{\tau,1/\varepsilon,b}(L_{\varepsilon,\tau}) \right\|_\infty \leq \delta_1(\tau, K, b), \tag{2.53}$$

where $\lim_{b\to 0} \limsup_{K\to\infty} \limsup_{\tau\to\infty} \delta_1(\tau, K, b) = 0$. (Of course, we just estimate $|\exp[-\xi] - \exp[-\eta]| \leq |\xi - \eta|$). Instead of spelling out the details for the above estimate which are easy, I want to give a comment on what is going on:

The one reason that we did cut out "only" $K\tau^{-1/(d-2)}$-holes was that we wanted to use a very crude bound of the total amount cut out. However, having now arrived at an approximation by $\Phi_{\tau,1/\varepsilon,K\tau^{-1/(d-2)}}(L_{\varepsilon,\tau})$, we want to cut bigger holes (for the procedure done in a moment). The reader might wonder that this is possible. The essential point is that the chance to hit the $a\tau^{-1/(d-2)}$-ball in an interval of length ε is very small anyway. Of course, we have to have to know this probability because we multiply it by τ in Φ, but it is not very important if our starting point is only close to the $a\tau^{-1/(d-2)}$-ball (still macroscopic) or "very close" (i.e. on scale $\tau^{-1/(d-2)}$). This region between "close" and "very close" is negligible, due to our lemma, essentially because we have the x-integration in the end.

Define now

$$\Phi_{\infty,\beta,b}(\mu) \stackrel{\text{def}}{=} \int_{\Lambda_N} dx \left(1 - \exp\left[-\beta\kappa_a \int \varphi_\varepsilon^b(y-x, z-x)\mu(dy,dz)\right]\right),$$

where $\varphi_\varepsilon^b(x,y)$ is $\varphi_\varepsilon(x,y)$ if x,y are both outside $B_b(0)$, and 0 otherwise. Lemma 2.63 b) now gives

$$\left\| \Phi_{\infty,1/\varepsilon,b}(L_{\varepsilon,\tau}) - \Phi_{\tau,1/\varepsilon,b}(L_{\varepsilon,\tau}) \right\|_\infty \leq \delta_2(\tau, b), \tag{2.54}$$

where $\lim_{\tau\to\infty} \delta_2(\tau, b) = 0$ for all b. Combining now (2.53) and (2.54), we get, by letting $\tau \to \infty$, $K \to \infty$, and finally $b \to 0$ (in this order):

$$\lim_{\tau\to\infty} \left\| \Phi_{\infty,1/\varepsilon,0}(L_{\varepsilon,\tau}) - E_\varepsilon(V_\tau) \right\|_\infty = 0.$$

$\Phi_{\infty,1/\varepsilon,0}(\mu)$ is continuous in μ, and therefore, we get the following large deviation principle for $E_\varepsilon(V_T)$ (ε arbitrary > 0, but fixed), which is based on a (strong) LDP for bivariate chains, stated after the proposition.

Proposition 2.64. $\{E_\varepsilon(V_\tau)\}_{\tau>0}$ *satisfies a* τ*-LDP with rate function*

$$J_\varepsilon(b) \stackrel{\text{def}}{=} \inf\left\{I_\varepsilon^{(2)}(\mu) : \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N),\ \Phi_{\infty,1/\varepsilon,0}(\mu) = b\right\}.$$

Here $I_\varepsilon^{(2)}(\mu)$ *is the rate function of the LDP for* $L_{\varepsilon,\tau}$ *on* $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$ *which is just*

$$I_\varepsilon^{(2)}(\mu) = \int \log\left(\frac{d\mu}{d(\mu_1 \otimes \pi_\varepsilon)}\right) d\mu,$$

if $\mu_1 = \mu_2$, μ_i *being the margins of* μ, *and* ∞ *otherwise.* π_ε *is the transition kernel of the Brownian on the torus on a time interval* ε.

The proposition follows from the considerations explained above, a contraction principle (see [43], Ch. III.5) and the following result

Theorem 2.65 (LDP for bivariate Markov chains). *Let* ξ_i, $i \in \mathbb{N}$ *be a Markov chain, taking values in some Polish space* S *with transition densities* $p(x,y)$ *with respect to a stationary measure* π *which satisfy*

$$1/C \le p(x,y) \le C.$$

Consider the bivariate empirical distribution

$$L_n^{(2)} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \delta_{(\xi_{i-1},\xi_i)}.$$

Then $\left(L_n^{(2)}\right)$ *satisfies a (strong)* N*-LDP in* $\mathcal{M}_1^+(S \times S)$ *with rate function*

$$I^{(2)}(\mu) \stackrel{\text{def}}{=} \begin{cases} \int \log \frac{d\mu}{d(\mu_1 \otimes p)} d\mu & \text{if } \mu_1 = \mu_2 \\ \infty & \text{if } \mu_1 \neq \mu_2 \end{cases},$$

where μ_1, μ_2 *are the two marginals of* μ. $\mu_1 \otimes p$ *is the measure* $\mu_1(dx)\,p(x,y)\,\pi(dy)$.

For a proof of this, see [43], Theorem IV.3.

We come now to the last step C) of the proof of Proposition 2.59. Up to now, we have a LDP for $E_\varepsilon(V_\tau)$, and we know that this quantity approximates the one we are interested in. We therefore only have to prove that the rate function approximates the right one. There is one delicacy. The rate function we have for fixed ε is a rate function of the bivariate chain. It is well known, that the rate function of the univariate discrete time ε-gap chain approximates the one for the Brownian motion as $\varepsilon \to 0$, and the rate function of the univariate discrete chain is the projection of the bivariate one. In our case, the function really depends on the bivariate chain. It however turns out that for small ε, the bivariate chain is essentially determined by the univariate one, up to a superexponential decay.

For $\mu \in \mathcal{M}_1^+(\Lambda_N)$, we write $I(\mu)$ for the standard large deviation rate function for the empirical distribution of the Brownian motion: $I(\mu) = \frac{1}{2}\int |\nabla g|^2\, dx$, $g^2(x) = \mu(dx)/dx$ if μ is absolutely continuous, and its density is in H_1 and $I(\mu) = \infty$ otherwise. We also denote by $I_\varepsilon : \mathcal{M}_1^+(\Lambda_N) \to [0,\infty]$ the projection of $I_\varepsilon^{(2)}$: $I_\varepsilon(\nu) = \inf\left\{ I_\varepsilon^{(2)}(\mu) : \mu_1 = \nu \right\}$. We collect some basic facts about these entropies which have been proved by Donsker and Varadhan or are simple consequences of their results:

Lemma 2.66. *Let $(\pi_t)_{t\geq 0}$ be the Brownian semigroup. Then for all $\nu, \mu \in \mathcal{M}_1^+(\Lambda_N)$ we have*

a) $I_\varepsilon(\nu) = -\inf_{u\in\mathcal{D}^+} \int \log \frac{\pi_\varepsilon u}{u} d\nu$, *where $\mathcal{D}^+$ is the set of positive measurable functions which are bounded and bounded away from* 0.
b) $t \to I_t(\nu)/t$ *is non-increasing with* $I(\nu) = \lim_{t\to 0} \frac{I_t(\nu)}{t}$.
c) $\|\nu - \nu\pi_s\|_{\mathrm{TV}} \leq 8\sqrt{I_s(\nu)}$ *for* $s > 0$
d $I_s(\nu\pi_t) \leq I_s(\nu)$ *for* $s, t > 0$.
e) $\|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} \leq 8\sqrt{I_s^{(2)}(\mu)}$

Proof. a) This is Theorem 2.1 of [33], combined with Lemma 2.1 of [32].

b) Let $u \in \mathcal{D}^+$ and $s, t > 0$. Then

$$\int \log \frac{\pi_{s+t} u}{u} d\nu = \int \log \frac{\pi_s(\pi_t)u}{\pi_t u} d\nu + \int \log \frac{\pi_t u}{u} d\nu \geq -I_s(\nu) - I_t(\nu).$$

Therefore $I_{s+t}(\nu) \leq I_s(\nu) + I_t(\nu)$. Hence, $I_t(\nu)/t$ is non-decreasing. The fact that $\lim_{t\to 0} I_t(\nu)/t = I(\nu)$ is Lemma 3.1 from [32].

c) This is Lemma 4.1 of [32]. (The function ϕ used there is easily seen to be $\leq 8\sqrt{x}$).

d) follows from the convexity of I_s.

e) Let $P^\mu(x, dy)$ be a transition kernel on Λ_N with $\mu = \mu_1 \otimes P^\mu$. Then

$$\|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} \leq \int \mu_1(dx)\, \|P^\mu(x,\cdot) - \pi_s(x,\cdot)\|_{\mathrm{TV}}\,.$$

By Theorem 4.1 of [28], we have

$$\|P^\mu(x,\cdot) - \pi_s(x,\cdot)\|_v \leq 8\sqrt{k(P^\mu(x,\cdot)|\pi_s(x,\cdot))},$$

where k is the usual Kullback–Leibler information, i.e.

$$k(\gamma|\sigma) = \int \log(d\gamma/d\sigma) d\gamma.$$

Therefore

$$\begin{aligned}\|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} &\leq 8 \int \mu_1(dx)\sqrt{k(P^\mu(x,\cdot)|\pi_s(x,\cdot))} \\ &\leq 8\sqrt{\int \mu_1(dx)k(P^\mu(x,\cdot)|\pi_s(x,\cdot))} = 8\sqrt{I_s^{(2)}(\mu)}.\end{aligned}$$

Next, we need an approximation of our functions $\Phi_{\infty,1/\varepsilon,0}$, for which we had derived a LDP by the Proposition 2.64, by the simpler functions $\Psi_\varepsilon : \mathcal{M}_1^+(\Lambda_N) \to [0,\infty)$, defined by

$$\Psi_\varepsilon(\nu) = \int dx \left[1 - \exp\left(-\frac{\kappa_a}{\varepsilon}\int_0^\varepsilon p_s(y-x)\nu(dy)\right)\right].$$

Lemma 2.67. *For any $K > 0$*

$$\lim_{\varepsilon\to 0} \sup_{\mu:\frac{1}{\varepsilon}I_\varepsilon^{(2)}(\mu)\leq K} \left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Psi_\varepsilon(\mu_1)\right| = 0.$$

Proof. We have $\Psi_\varepsilon(\mu_1) = \Phi_{\infty,1/\varepsilon,0}(\mu_1 \otimes \pi_\varepsilon)$, and therefore

$$\begin{aligned}&\left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Psi_\varepsilon(\mu_1)\right| \\ &= \left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Phi_{\infty,1/\varepsilon,0}(\mu_1 \otimes \pi_\varepsilon)\right| \\ &\leq \frac{\kappa_a}{\varepsilon}\left|\int dx \int_{\Lambda_N\times\Lambda_N} \varphi_\varepsilon(y-x,z-x)\left(\mu(dy,dz) - \mu_1\otimes\pi_\varepsilon(dy,dz)\right)\right| \\ &\leq \frac{\kappa_a}{\varepsilon}\int dx \int_{\Lambda_N\times\Lambda_N} \varphi_\varepsilon(y-x,z-x)\left|\mu - \mu_1\otimes\pi_\varepsilon\right|(dy,dz) \\ &= \kappa_a \left\|\mu - \mu_1\otimes\pi_\varepsilon\right\|_{TV}.\end{aligned}$$

The Lemma follows now from Lemma 2.66 e).

Next, we define $\Gamma : L_1^+(\Lambda_N) \to [0,\infty)$ by

$$\Gamma(f) = \int dx \left[1 - \exp\left(-\kappa_a f(x)\right)\right].$$

Lemma 2.68. *For any $K > 0$*

$$\lim_{\varepsilon\to 0} \sup_{\nu:\frac{1}{\varepsilon}I_\varepsilon(\nu)\leq K} \left|\Gamma\left(\frac{d\nu}{dx}\right) - \Psi_\varepsilon(\nu)\right| = 0.$$

(Remark that if $I_\varepsilon(\nu)$ is finite, then $d\nu \ll dx$)

Proof.

$$\left|\Gamma\left(\frac{d\nu}{dx}\right) - \Psi_\varepsilon(\nu)\right|$$

$$
\begin{aligned}
&\leq \int dx \left| \exp\left(-\frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \int p_s(y-x)\nu(dy)\right) - \exp\left(-\frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \frac{d\nu}{dx}(x)\right)\right| \\
&\leq \int dx \frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \left|\frac{\nu\pi_s}{dx}(x) - \frac{d\nu}{dx}(x)\right| = \frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \left\|\nu\pi_s - \nu\right\|_{\mathrm{TV}} .
\end{aligned}
$$

Now, for $s \leq \varepsilon$

$$
\begin{aligned}
\|\nu\pi_s - \nu\|_{\mathrm{TV}} &\leq \|\nu\pi_s\pi_\varepsilon - \nu\pi_s\|_{\mathrm{TV}} + \|\nu\pi_{s+\varepsilon} - \nu\|_{\mathrm{TV}} \\
&\leq 8\sqrt{I_\varepsilon(\nu\pi_s)} + 8\sqrt{I_{\varepsilon+s}(\nu)}.
\end{aligned}
$$

Now $I_\varepsilon(\nu\pi_s) \leq I_\varepsilon(\nu)$ by Lemma 2.66d).

Furthermore, $I_{\varepsilon+s}(\nu) \leq 2\varepsilon I_{\varepsilon+s}(\nu)/(\varepsilon+s) \leq 2I_\varepsilon(\nu)$ by Lemma 2.66b). Therefore, we get $\|\nu\pi_s - \nu\|_{\mathrm{TV}} \leq 8(1+\sqrt{2})\sqrt{K\varepsilon}$ if $I_\varepsilon(\nu) \leq K\varepsilon$. Using this, the Lemma follows.

We can now finally finish the proof of Proposition 2.59.

Consider a continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. Then

$$
\begin{aligned}
&\lim_{\tau\to\infty} \frac{1}{\tau} \log E\left(\mathrm{e}^{\tau f(V_\tau)}\right) \\
&= \lim_{\varepsilon\to 0}\lim_{\tau\to\infty} \frac{1}{\tau}\log E\left(\exp\left[\tau f(E_\varepsilon(V_\tau))\right]\right) \text{ (Proposition 2.61)} \\
&= \lim_{\varepsilon\to 0}\sup_{\mu}\left\{f(\Phi_{\infty,1/\varepsilon,0}(\mu)) - \frac{1}{\varepsilon}I_\varepsilon^{(2)}(\mu)\right\} \text{ (Proposition 2.64)} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0}\sup_{I_\varepsilon^{(2)}(\mu)\leq\varepsilon K}\left\{f(\Phi_{\infty,1/\varepsilon,0}(\mu)) - \frac{1}{\varepsilon}I_\varepsilon^{(2)}(\mu)\right\} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0}\sup_{I_\varepsilon^{(2)}(\mu)\leq\varepsilon K}\left\{f(\Psi_\varepsilon(\mu_1)) - \frac{1}{\varepsilon}I_\varepsilon^{(2)}(\mu)\right\} \text{ (Lemma 2.67).}
\end{aligned}
$$

We now use that I_ε is the projection of $I_\varepsilon^{(2)}$, namely

$$
I_\varepsilon(\nu) = \inf\left\{I_\varepsilon^{(2)}(\mu) : \mu_1 = \nu\right\}.
$$

Therefore

$$
\begin{aligned}
\lim_{\tau\to\infty}\frac{1}{\tau}\log E\left(\mathrm{e}^{\tau f(V_\tau)}\right) &= \lim_{K\to\infty}\lim_{\varepsilon\to 0}\sup_{I_\varepsilon(\nu)\leq\varepsilon K}\left\{f(\Psi_\varepsilon(\nu)) - \frac{1}{\varepsilon}I_\varepsilon(\nu)\right\} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0}\sup_{I_\varepsilon(\nu)\leq\varepsilon K}\left\{f\left(\Gamma\left(\frac{d\nu}{d\lambda}\right)\right) - \frac{1}{\varepsilon}I_\varepsilon(\nu)\right\} \\
&= \sup_\nu\left\{f\left(\Gamma\left(\frac{d\nu}{d\lambda}\right)\right) - I(\nu)\right\},
\end{aligned}
$$

the second equation by Lemma 2.68. This proves now the Proposition 2.59 by applying the “inverse” of Varadhan’s lemma, also called Bryc’s lemma (see [43], p. 33).

2.7 Remarks on the polaron problem

A problem similar in spirit to the considerations in this chapter is connected with the so-called polaron problem. The questions on path measures are mathematically open, but I explain what is known and what is conjectured in this interesting problem. The physical problem is coming from a quantum mechanical discussion of a charged particle, e.g. an electron, which is moving in a crystal whose lattice sites can be polarized. The electron is then dragging around it a cloud of polarized lattice points which will influence its behavior. In particular, the electron moves as having a different mass. This is the so-called effective mass. I will not discuss the physical background for which I refer to the Lectures of Feynman [39]. Feynman gave a path integral formulation of the problem, and questions about the effective mass can be formulated in terms of a path measure obtained from the Brownian path by a self-attracting interaction. We are describing actually only the Fröhlich polaron (after the solid-state physicist H. Fröhlich). There are other ones which not all fit into this framework. Here is the path measure, which has two parameters $\beta, \lambda > 0$:

$$\widehat{P}_{T,\beta,\lambda}\left(d\omega\right) \stackrel{\text{def}}{=} \frac{1}{Z_{T,\beta,\lambda}} \exp\left[\frac{\beta\lambda}{2}\int_0^T ds \int_0^T dt\, \frac{\mathrm{e}^{-\lambda|t-s|}}{|\omega_t - \omega_s|}\right] P\left(d\omega\right).$$

P is the law of the three-dimensional Brownian motion. The parameter β is not of importance and can be scaled away. So we put $\beta = 1$. The above form is actually not exactly the one given in Feynman, but it follows by some trivial rescaling. The parameter λ is $1/\alpha^2$, where α is the "physical" coupling parameter between the electron and the lattice points. We are interested in the case $\lambda \to 0$, which corresponds to the strong coupling limit for the physical problem.

First remark that the interaction is self-attracting: The new measure favors paths which, at least on short time scales, are clumping together. (It should also be remarked that the integral is well defined in three dimensions, despite of the Coulomb singularity). An important feature is that the interaction decays exponentially in $|t-s|$, so that for fixed λ, the interaction is essentially short range in time. The effect is that the path measure behaves essentially diffusive (for fixed λ), i.e. after Brownian rescaling, $\widehat{P}_{T,\lambda}$ converges to the Brownian motion with a rescaled diffusion coefficient $D(\lambda) > 0$, i.e.

$$\lim_{T\to\infty} \widehat{P}_{T,\lambda}\rho_T^{-1} = P^{(D(\lambda)I)},$$

where $P^{(\Sigma)}$ is the Brownian motion with covariance matrix Σ, and $\rho_T(\omega) \stackrel{\text{def}}{=} \omega(T\,\cdot)/\sqrt{T}$, as usual. There does not seem to exist a proof of this for the above model. Spohn ([68]) proved it under some smoothness assumption on the interaction (not including the Coulomb singularity, however), but I think there can be no serious doubt that the statement is correct, despite the fact that there had been speculations in the physics literature about a roughening transition.

An interesting problem is to determine the path properties of the above path measure for $\lambda \sim 0$, and in particular to determine $D(\lambda)$, but this is mathematically open. The diffusion constant $D(\lambda)$ is directly related to the effective mass (see [68]).

An easier problem is to determine the (rough) behavior of the partition function $Z_{T,\lambda}$, and this has been done in a celebrated result by Donsker and Varadhan. The argument roughly is that for λ small, the interaction, despite being short range, gets more and more smeared out, and one might guess that for $\lambda \to 0$ (after $T \to \infty$), the behavior of Z does not much deviate from the situation where one would take the mean-field model with a Hamiltonian

$$\frac{1}{T}\int_0^T ds \int_0^T dt \, \frac{1}{|\omega_t - \omega_s|}.$$

This was the content of the Pekar-conjecture and is true:

Theorem 2.69 (Donsker-Varadhan (see [36]).

$$\begin{aligned}\lim_{\lambda\to 0}\lim_{T\to\infty}\frac{1}{T}\log Z_{T,\lambda} &= \lim_{T\to\infty}\frac{1}{T}\log E\left(\exp\left[\frac{1}{T}\int_0^T ds\int_0^T dt\,\frac{1}{|\omega_t-\omega_s|}\right]\right)\\ &= \sup_{\|g\|_2=1}\left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{dx\,dy}{|x-y|}g^2(x)\,g^2(y)-\frac{1}{2}\|\nabla g\|_2^2\right\}.\end{aligned}$$

The second equation is in the spirit of the large deviation arguments we have used above: The mean-field Hamiltonian can be written as

$$\frac{1}{T}\int_0^T ds\int_0^T dt\,\frac{1}{|\omega_t-\omega_s|} = T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}L_T(dx)\,L_T(dy),$$

from which the last equality "follows" by the Donsker-Varadhan Theorem 2.17 stated in the introduction 2.1, but there are a number of technical difficulties, e.g. the singularity. However, the the main difficulty was to prove the first equation for which Donsker and Varadhan developed their "level-3" large deviation principles. I will not go into that here. The variational problem above had been discussed in a celebrated paper by Lieb [50] who proved that there are unique maximizers modulo shifts.

From this result it remains somewhat unclear what to expect for the path measure. It should however be clear that the problem is in some way related to the mean-field path measure:

$$\widehat{P}_T^{\mathrm{mf}}(d\omega) \stackrel{\mathrm{def}}{=} \frac{1}{Z_T^{\mathrm{mf}}}\exp\left[\frac{1}{T}\int_0^T ds\int_0^T dt\,\frac{1}{|\omega_t-\omega_s|}\right]P(d\omega).$$

The following "result" is then in complete analogy to our Theorem 2.34:

Conjecture 2.70.

$$\lim_{T\to\infty} \widehat{P}_T^{\mathrm{mf}} L_T^{-1} = \frac{\int g(x)\ \delta_{\theta_x g^2}\, dx}{\int g(x)\ dx},$$

where g is the unique L_2-normed, positive, rotational symmetric solution of the variational problem in Theorem 2.69. (By an abuse of notation, $\theta_x g^2$ stands for the measure on $\mathbb{R}^3$ with this density).

This has also not been proved, but in my view there can be no serious doubt that it is correct. A proof could probably be given along the lines of the proof of Theorem 2.34, with some additional technical problems coming from the Coulomb singularity.

Given this "result", one should believe that our real path measure $\widehat{P}_{T,\lambda}$ looks at least on scales of order $1/\lambda$ such that the local empirical measures are close to some $\theta_x g^2$. Based on this, Herbert Spohn in [68] gave a heuristic derivation of what $D(\lambda)$ should be. This heuristic argument is based on a number of very simplifying assumptions. First chop the time axis into pieces $I_k \overset{\mathrm{def}}{=} [(k-1)\,\eta, k\eta)$ of length $1 \ll \eta \ll 1/\lambda$, $1 \leq k \leq T/\eta$, and consider the empirical distribution on each of these time slots:

$$L_{k,T} \overset{\mathrm{def}}{=} \frac{1}{\eta} \int_{I_k} \delta_{\omega_s} ds.$$

The basic assumption is that these local empirical measures are close to some shift of g^2 :

$$L_{k,T} \sim g^2 (\cdot - \theta_k),$$

which is certainly plausible if one believes in the above Conjecture 2.70. Spohns argument is now based on the following further assumptions:

- The only relevant information are these θ_k and the fluctuations of $L_{k,T}$ around $g^2(\cdot - \theta_k)$ are not playing any rôle.
- The fact that the end points of the Brownian motion at the end of one of the time slots is the same as at the beginning of the next slot is not having any influence.
- The a priori distribution of the θ_k is "uniform distribution" on $\mathbb{R}^3$.
- The diffusion constant can be evaluated (in the $\lambda \to 0$ limit) by forgetting anything except the θ_k, and expand the Hamiltonian in terms of these parameters. This evidently leads to a Gaussian theory for the sequence $(\theta_k)_{k\geq 1}$.

The rest of the argument is then plain sailing, but it goes without saying that a justification of the above assumptions is very far from obvious, and the reader will probably have serious doubts that the answer obtained in this way is correct (as had I, first). First, we can write the Hamiltonian as

$$
\begin{aligned}
&\frac{\lambda}{2}\sum_{k,l=1}^{T/\eta}\int_{I_k} ds \int_{I_l} dt\, \mathrm{e}^{-\lambda|t-s|}\frac{1}{|\omega_s-\omega_t|}\\
&=\frac{\lambda}{2}\sum_{k,l=1}^{T/\eta}\mathrm{e}^{-\lambda\eta|l-k|}\int_{I_k} ds \int_{I_l} dt\, \frac{1}{|\omega_s-\omega_t|}\\
&\approx\frac{\lambda\eta^2}{2}\sum_{k,l=1}^{T/\eta}\mathrm{e}^{-\lambda\eta|l-k|}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \frac{1}{|x-y|}g^2(x-\theta_k)g^2\left(y-\theta_l\right),
\end{aligned}
$$

where we have used the basic assumption that we can switch from the local empirical measures to the $g^2(x-\theta_k)$. Expanding now in terms of $\theta_k-\theta_l$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \frac{1}{|x-y|}g^2(x-\theta_k)g^2\left(y-\theta_l\right)\\
&=\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \frac{1}{|x-y-(\theta_k-\theta_l)|}g^2(x)g^2\left(y\right)\\
&\approx\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \frac{1}{|x-y|}g^2(x)g^2\left(y\right)-4\pi\left\|\theta_k-\theta_l\right\|^2\int_{\mathbb{R}^3} g^4\left(x\right)dx.
\end{aligned}
$$

The first part is of no relevance as it cancels with the normalization, so we just have to look at the Gaussian measure

$$
\frac{1}{Z}\exp\left[-\frac{4\pi\lambda\eta^2}{2}\sum_{k,l=1}^{T/\eta}\mathrm{e}^{-\lambda\eta|l-k|}\left\|\theta_k-\theta_l\right\|^2\int_{\mathbb{R}^3} g^4\left(x\right)dx\right].
$$

At the beginning, the path measure is tied down, so that we put $\theta_1\approx 0$, and consider that as a density for $\theta_2,\ldots,\theta_{T/\eta}$. Then $\theta_{T/\eta}$ has covariance matrix

$$
\begin{aligned}
&\approx\frac{T}{\eta}\left[4\pi\lambda\eta^2\int_{\mathbb{R}^3} g^4\left(x\right)dx\sum_{k=0}^{\infty}k^2\mathrm{e}^{-\lambda\eta k}\right]^{-1} I\\
&\approx\frac{T}{\eta}\left[\frac{4\pi}{\lambda^2\eta}\int_{\mathbb{R}^3} g^4\left(x\right)dx\int_0^{\infty}x^2\mathrm{e}^{-x}dx\right]^{-1} I=\frac{T\lambda^2}{8\pi\int_{\mathbb{R}^3} g^4\left(x\right)dx}I.
\end{aligned}
$$

Therefore,

$$
D\left(\lambda\right)=\frac{\lambda^2}{8\pi\int_{\mathbb{R}^3} g^4\left(x\right)dx}+o\left(\lambda^2\right),
$$

as $\lambda\to 0$.

This is of course very far from a mathematical proof. The Spohn-heuristics has however recently been verified in a much simpler case, namely for a one-dimensional plane rotator model with a Kac-type interaction by Petermann [61]. Petermann does not use large deviation theory, but relies on the Griffiths inequalities which cannot be applied to the polaron, but the result gives some confidence that the result should be correct.

3. One-dimensional pinning-depinning transitions

I present in this chapter two results on one-dimensional random walks interacting with a layer. This layer, for the random walk, is just the path identical to zero. The interaction presented in the two sections are slightly different, but the effects are quite similar. The first section discusses what in physics literature is called a wetting transition. Here the layer is acting as a hard wall in the sense that the random walk has to stay on one side, but there is also an attractive interaction between the random walk and the wall. There is a considerable literature around such wetting-transitions, an I present here only the very most simple case where such a transition occurs. Of considerable interest are cases where the random walk is replaced by a random surface, and then, of course, similar questions for more complicated random interfaces, like interfaces in Ising type models, but I leave this out.

In the second section I discuss a pinning-depinning transition for a model where the interaction with the "wall" is produced by random components of the random walk, which make the "nodes" of the random walk either be inclined to be on the positive side of the wall, or on the negative. This is a very simple model of a polymer chain whose components are either "oil repellent" or "water repellent", and which is placed at an interface between water below and oil above. The components on the polymer chain are randomly placed, and one would like to know what the effect is.

3.1 Wetting transition for a one dimensional random walk

We consider a standard random walk in one dimension (discrete time), i.e. the probability measure P_n on

$$\Omega_n \stackrel{\mathrm{def}}{=} \{\omega = (\omega_0 = 0, \omega_1, \ldots, \omega_n) : |\omega_i - \omega_{i-1}| = 1 \text{ for } i = 1, \ldots, n\},$$

gives equal weight 2^{-n} to all the paths. It will be convenient in this section, although it is not important at all, to work with a tied down walk. Therefore, let

$$\Omega_{2n}^0 \stackrel{\mathrm{def}}{=} \{\omega \in \Omega_{2n} : \omega_{2n} = 0\},$$

and again, P^0_{2n} is the uniform distribution. Let $\beta > 0$ as usual be a positive coupling parameter. We first discuss only the case with an attraction to the "wall" $(0, \ldots, 0)$. So we define

$$\hat{P}_{2n,\beta}(\omega) = \frac{1}{Z_{2n,\beta}} \exp\left[\beta \sum_{i=1}^{2n-1} 1_{\omega_i=0}\right] P^0_{2n}(\omega).$$

This is an extremely simple object, much simpler, of course, than the models discussed in Chapter 2. We prove that it localizes for every positive $\beta > 0$:

Proposition 3.71. *For any $\beta > 0$*

a)

$$f(\beta) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{2n} \log Z_{2n,\beta} = -\frac{1}{2} \log\left(1 - \mathrm{e}^{-2\beta}\right) > 0.$$

b) *There exist $m(\beta) > 0$, and $A(\beta)$ such that*

$$\hat{E}_{2n,\beta}(\omega_i\omega_j) \le A(\beta) \exp(-m(\beta)|i-j|).$$

Proof. a) is quite easy. Let $q(l)$, $l \in 2\mathbb{N}$, be the distribution of the standard first return time to 0 :

$$q(2l) \stackrel{\text{def}}{=} P_{2l}\left(\omega_j \neq 0,\ 1 \le j \le 2l-1,\ \omega_{2l} = 0\right).$$

The exact distribution is well known, and the generative function is

$$\sum_l z^l q(l) = 1 - \sqrt{1 - z^2},\ z < 1.$$

Furthermore

$$Z_{2n,\beta} = \frac{\sum_{m\ge1} \sum_{0=l_0<l_1<\ldots l_{m-1}<l_m=n} \prod_{j=1}^m q(l_j - l_{j-1})}{P_{2n}(\omega_{2n} = 0)}.$$

It is well known that the denominator is of order $1/\sqrt{n}$, so we only have to take care of the numerator, call it $\widetilde{Z}_{2n,\beta}$. If $0 \le \lambda < \sqrt{1 - \mathrm{e}^{-2\beta}}$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda^{2n} \widetilde{Z}_{2n,\beta} &= \sum_{m=1}^{\infty} \mathrm{e}^{\beta(m-1)} \left(\sum_{n=1}^{\infty} q(2n)\,\lambda^{2n}\right)^m \\ &= \frac{1 - \sqrt{1-\lambda^2}}{1 - \mathrm{e}^{\beta}\sqrt{1-\lambda^2}},\end{aligned}$$

and this diverges for $\lambda \uparrow \sqrt{1 - \mathrm{e}^{-2\beta}}$. From this, the exact form of $f(\beta)$ follows.

We next prove that a) implies b), which is very simple, too.

Consider the random set Γ_{2n} of “dry” points in $\{1,\dots,2n-1\}$, i.e. the points where the random walk visits 0. Clearly, if

$$A = \{\ell_1,\dots,\ell_{m-1}\},\ 0=\ell_0<\ell_1<\ldots<\ell_{m-1}<\ell_m=2n,$$

then

$$P(A) = \prod_{j=1}^{m} f(\ell_j-\ell_{j-1}).$$

We set $\varrho^{[0,2n]}(A) \stackrel{\mathrm{def}}{=} \hat{P}_{2n,\beta}(\Gamma_{2n}=A)$. Then

$$\varrho^{[0,2n]}(A) = P_{2n}(\Gamma_{2n}=A)e^{\beta(m-1)}/Z_{2n}^{\beta}.$$

Conditionally on $\{\Gamma_{2n}=A\}$, both P_{2n}^{0} and $\hat{P}_{2n,\beta}$ have exactly the same law on paths just by choosing the excursions independently on all the intervals $[\ell_{i-1},\ell_i]$. If $i\geq j$ and $A\cap[i,j]\neq\emptyset$, then ω_i and ω_j are independent under this conditioned law. We therefore get

$$\hat{E}_{2n,\beta}(\omega_i\omega_j) = \sum_{A:A\cap[i,j]=\emptyset} \varrho^{[o,2n]}(A)E_{2n}^{0}(\omega_i\omega_j|\Gamma_{2n}=A). \tag{3.1}$$

If $0\leq k<\ell\leq 2n$, we denote by $\mathcal{M}_{k,\ell}$ the set of subsets A of $\{1,\dots,2n-1\}$ such that $A\cap(k,\ell)=\emptyset$, $A\cup\{0,2n\}\supset\{k,\ell\}$.If $A\in\mathcal{M}_{k,\ell}, k<i<j<\ell$, then

$$|E_{2n}^{0}(\omega_i\omega_j|\Gamma_{2n}=A)|\leq \ell-k. \tag{3.2}$$

Remark also that for $A\in\mathcal{M}_{k,\ell}, B\subset(k,\ell)$

$$\varrho^{[0,n]}(A\cup B) = \frac{\varrho^{[0,n]}(A)\varrho^{[k,\ell]}(B)Z_{\ell-k,\beta}}{f(\ell-k)}.$$

Therefore

$$\sum_{A\in\mathcal{M}_{k,\ell}} \varrho^{[0,n]}(A) = \sum_{A\in\mathcal{M}_{k,\ell}}\sum_{B\subset(k,\ell)} \varrho^{[0,n]}(A)\varrho^{[k,\ell]}(B) \leq \frac{f(\ell-k)}{Z_{\ell-k,\beta}}$$

Using this together with a) and (3.1) and (3.2) proves b).

We now change the model in a way which leads to an interesting localization-delocalization transition similar to the one encountered in Section 1. This had been observed by M. Fisher [40] in his Boltzmann Lectures. We just replace the ordinary random walk by one having a “hard wall” condition, meaning that the walk has to stay positive. Therefore, we consider

$$\Omega_{2n}^{+} = \{\omega\in\Omega_{2n}^{0} : \omega_i\geq 0\,\forall i\},$$

and P_{2n}^+ be the uniform distribution on Ω_{2n}^+. The large n behavior of P_{2n}^+ is well known: After Brownian scaling, the path measure converges to that of a Brownian excursion on a fixed time interval. In particular, $E_{2n}^+(\omega_n)$ is of order $\sqrt{n}$. Taking a coupling parameter $\beta > 0$ we again introduce an attractive "polymer-to-wall" interaction by defining

$$\hat{P}_{2n,\beta}^+(\omega) = \frac{1}{Z_{2n}^{+,\beta}} P_{2n}^+(\omega) \exp\left[\beta \sum_{j=1}^{2n-1} 1_{\omega_j=0}\right].$$

One then has the following result

Theorem 3.72. *(M. Fisher [40]). There exists $\beta_{cr} > 0$ such that*

1. *For $\beta < \beta_{cr}$*
   a) $f^+(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_{2n,\beta}^+ = 0$
   b) *The path measure converges after Brownian scaling to that of Brownian excursion.*
2) *For $\beta > \beta_{cr}$*
   a) $f^+(\beta) > 0$
   b) *There exist $A(\beta), m(\beta) > 0$ such that*

$$\mathrm{cov}_{\hat{P}_{2n,\beta}^+}(\omega_i, \omega_j) \leq A(\beta) \exp(-m(\beta)|i-j|)$$

uniformly in n, i, j.

Proof. I prove here only the existence of β_{cr} such that $f^+(\beta) = 0$ for $\beta \leq \beta_{cr}$ and $f^+(\beta) > 0$ for $\beta > \beta_{cr}$. The exponential decay in 2(b) is more delicate than in Proposition 3.71. There is a recent paper [45] where these properties are proved in details, covering even the critical case $\beta = \beta_{cr}$.

$f^+(\beta)$ is evidently ≥ 0, convex in $\beta \geq 0$, and satisfies $f^+(0) = 0$. It is also very easy to see that $f(\beta) > 0$ for large enough β (just estimate $Z_{2n,\beta}^+$ from below by taking the single path which hits 0 at all even times). Therefore, the only issue is to prove that $f^+(\beta) = 0$ for small enough $\beta > 0$. To do this, remark first that

$$Z_{2n,\beta}^+ = \frac{\sum_m \sum_{0<\ell_1<\ldots<\ell_{m-1}<2n} 2^{-m} \mathrm{e}^{\beta(m-1)} \prod_{j=1}^m q(\ell_j - \ell_{j-1})}{P_{2n}^0(\Omega_{2n}^+) P_{2n}(\omega_{2n} = 0)}.$$

It is well known that $P_{2n}^0(\Omega_{2n}^+) \sim 1/n$, and therefore, we only have to take care of the numerator, call it again $\tilde{Z}_{2n,\beta}^+$. If $0 \leq \lambda < 1$, then one has now,

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda^{2n} \tilde{Z}_{2n,\beta}^+ &= \sum_{m=1}^{\infty} 2^{-m} \mathrm{e}^{\beta(m-1)} \left(\sum_{n=1}^{\infty} f(2n)\lambda^{2n}\right)^m \\ &= \frac{1}{2}(1 - \sqrt{1-\lambda^2}) \left[1 - \frac{\mathrm{e}^\beta}{2}(1 - \sqrt{1-\lambda^2})\right]^{-1}.\end{aligned}$$

Evidently, if $\beta < \log 2$, then this is finite for all $\lambda < 1$, and therefore $f^+(\beta) = 0$.

With this, I close this short discussion on wetting transitions. There are much more refined results. There are two recent papers [45] and [38], treating this in much more details. As remarked already, there also had been considerable work on such wetting transitions for higher-dimensional surfaces, instead of the one-dimensional "random string". Proofs for such surfaces are generally much more delicate, and many questions which are relatively easy for stings become much more delicate for surfaces. I don't want to enter this topic here, but refer to some of the relevant recent papers [16], [27], [31], [44], [17].

3.2 A heteropolymer near an interface

We consider in this section an interesting interaction of the random walk (the "heteropolymer") with the wall which is produced by a random environment which is given by a sequence of i.i.d. random variables $\sigma_i = \pm 1,\ 1 \le i < \infty$, which are also independent of the random walk. We denote the law of the σ_i by $\mathbb{P}$, but we will allow that the σ_i are asymmetric: $\mathbb{P}(\sigma_i = 1) = (1+h)/2$, and we write then $\mathbb{P}_h$. Remark that $\mathbb{E}_h(\sigma_i) = h$. We always assume that $h \ge 0$, the other case being symmetric. Fixing an environment $\sigma \in \{-1,1\}^{\mathbb{N}}$, we consider a transformed path measure on Ω_n by

$$\widehat{P}_{n,\beta,\sigma}(\omega) \stackrel{\text{def}}{=} 2^{-n} \exp\left[\beta \sum_{i=1}^{n} \sigma_i \Delta_i(\omega)\right] \Big/ Z_{n,\beta,\sigma}$$

where $\beta \ge 0$ is the usual coupling parameter, $\Delta_i(\omega) =$sign(ω_i), (sign$(0) \stackrel{\text{def}}{=} 0$), and

$$Z_{n,\beta,\sigma} = \sum_{\omega \in \Omega_n} 2^{-n} \exp\left[\beta \sum_{i=1}^{n} \sigma_i \Delta_i(\omega)\right].$$

We are interested in the behavior of $\widehat{P}_{n,\beta,\sigma}$ for large n which hold almost surely with respect to to $\mathbb{P}_h$.

This model has been introduced in the physics literature as a very simplified model for a so called heteropolymer. Think of (i,ω_i) as a (directed) polymer on $\mathbb{Z}^2$, modelling a polymer chain whose one end $(0,0)$ is attached at a border of two "liquids", the heavier one (say water) occupying $(i,j) \in \mathbb{Z}^2,\ j < 0$, and the lighter one (say oil) the upper space $(i,j),\ j > 0$. We think now that our polymer chain $(i,\omega_i), i \ge 0$, is composed of molecules which are either water-repellent, meaning $\sigma_i = 1$, or oil repellent, i.e. $\sigma_i = -1$, but we allow that they do not appear in equal amounts, and we assume that they are randomly placed. The polymer then gets larger weight when the "energy" $-\sum \sigma_i \Delta_i$ is low.

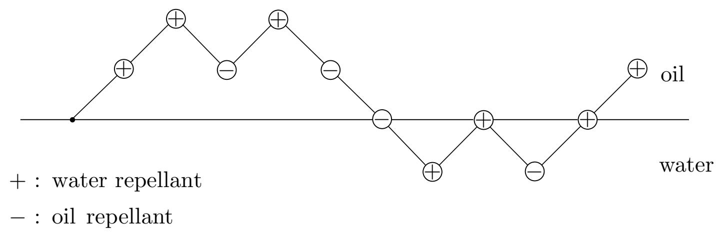


Fig. 3.1.

The model has been introduced in the physics literature [41], [42], and a mathematically rigorous treatment started with the paper by Sinai [65], and then in [15], [66], [4] and most recently by Biskup [8]. The paper [15] treats actually a slightly different model, where the σ_i are given as $h+\sigma_i^0$, and where the σ_i^0 are symmetric. This is technically slightly more convenient at some places. There are also recent papers in the physics literature, see e.g. [60]. The main question was to discuss the localization-delocalization behavior of the model. Evidently, the random walk (ω_i) has essentially two basic strategies to get its energy lowered. One is just to hang around the "water-oil" interface and switching to the right side as often as possible. This would mean that the polymer gets localized near the interface. It is however not evident that this should be the dominant effect even in the case $h = 0$, for arbitrary $\beta > 0$. However, this is exactly what Sinai has proved. If $h > 0$, i.e. when the water repellent nodes dominate, then the path could just stay on the upper half, getting dominant satisfaction cheaply, and it is not clear if this is better or worse than to act more sophisticated by hanging around the interface, and dipping from time to time into the watery side in order to please the now fewer oil repellent nodes .

The basic result in [15] is that for any $h > 0$, there is a transition from a delocalized region, when β is small, where it is essentially not worthwhile for the path to chase after satisfaction for the minority of oil-repellent guys, and a large β region where this is better than to stay lazily in the oil all the time.

I have to add here that we did not prove in [15] that the path measure really behaves in this way, but we worked instead purely with free energy considerations. This is the same as the evaluation in Chapter 2 of the rough leading order asymptotics which suggests but does not prove how the path measures really behaves. I hope it had become clear in chapter that to determine really the behavior of the path measure is usually then still a different story.

Proposition 3.73. a) *For any $\beta \geq 0, 0 \leq h \leq 1$*

$$\Phi(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,\beta,\sigma}$$

exists $\mathbb{P}_h$*-a.s. and in* $L_1(\mathbb{P}_h)$ *and does not depend on* σ.

b) Φ *is convex as a function of* β

c) $\beta h \leq \Phi(\beta, h) \leq \beta$.

d) Φ *is jointly continuous in* (β, h).

Proof. a) is a standard application of the subadditive ergodic theorem, and I refer to [15]. Remark that from L_1-convergence, one has

$$\Phi(\beta, h) = \lim_{n\to\infty} \mathbb{E}_h \log E \exp\left[\beta \sum_{i=1}^n \sigma_i \Delta_i\right].$$

b) This is the standard application using the Hölder inequality

$$\begin{aligned} &E \exp\left[\lambda\beta \sum_{i=1}^n \sigma_i \Delta_i + (1-\lambda)\beta \sum_{i=1}^n \sigma_i \Delta_i\right] \\ &\leq E\left(\exp \beta \sum_{i=1}^n \sigma_i \Delta_i\right)^{\lambda} \left(E \exp \sum_{i=1}^n \sigma_i \Delta_i\right)^{1-\lambda}. \end{aligned}$$

c) The upper bound is trivial, and the lower bound nearly so, but it is of crucial importance for what follows:

$$\begin{aligned} E \exp\left[\beta \sum_{i=1}^n \sigma_i \Delta_i\right] &\geq E\left(\exp\left[\beta \sum_{i=1}^n \sigma_i \Delta_i\right]; \omega_i > 0, 1 \leq i \leq n)\right) \\ &= \exp\left[\beta \sum_{i=1}^n \sigma_i\right] P(\omega_i > 0, 1 \leq i \leq n). \end{aligned}$$

The second factor is well known to be of order $1/\sqrt{n}$, and therefore plays no rôle for the free energy. So the bound follows.

d) is fairly evident from the fact that $|\sum_{i=1}^n \sigma_i \Delta_i| \leq n$. If $h_1 < h_2$, we couple $\sigma^1 = (\sigma_i^1)$ with law $\mathbb{P}^{h_1}$, and σ^2 with law $\mathbb{P}^{h_2}$ in the usual way such that $\sigma_i^1 \leq \sigma_i^2$ for all i. Then

$$\left|\sum_{i=1}^n \sigma_i^1 \Delta_i - \sum_{i=1}^n \sigma_i^2 \Delta_i\right| \leq \sum_{i=1}^n (\sigma_i^2 - \sigma_i^1),$$

and using that this is $\leq 2hn$ eventually, w.p. 1, the (Lipshitz) continuity follows.

From the proof of the lower bound in c), it is plausible that the path stays localized if $\Phi(\beta, h) > \beta h$. That this is actually the case had been proved for $h = 0$ by Sinai (who did not prove $\Phi(\beta, 0) > 0$), and in the full generality where $\Phi(\beta, h) > \beta h$ by Biskup [8]. The case $\Phi(\beta, h) = \beta h$ means that there is nothing better the path can do than just essentially to stay positive, at least not on an exponential scale. It is natural to expect that at least in the interior of this region, this means that the path measure $\hat{P}_n^{\beta,\sigma}$ except perhaps for a few steps at the beginning has to concentrate on the "oily" paths. To be precise, we suspect that the following is true:

Conjecture 3.74. Assume that (β, h) is such that for some $\varepsilon > 0$ (which may depend on (β, h)) one has $\Phi(\beta, h') = \beta h'$ for $h \geq h' - \varepsilon$. Then

$$\lim_{K\to\infty} \sup_n \hat{P}_n^{\beta,\sigma}(\max\{i : \omega_i \leq 0\} \geq K) = 0$$

$\mathbb{P}_h$ - a.s.

This would imply that the path, after Brownian rescaling has as limit distribution the Brownian meander $\mathbb{P}_h$-a.s. The above conjecture looks very natural, there seem however to be considerable difficulties to prove it.

For the rest of this section, we focus entirely on the behavior of $\Phi(\beta, h)$.

Our first task is to prove that a "delocalized" phase, i.e. a region where $\Phi(\beta, h) = \beta h$, really exists. As $\Phi(\beta, h) \geq \beta h$, this amounts to prove an upper bound for Φ.

One of the basic tricks in the business of random media is to try to prove that the "quenched" free energy, i.e. our $\Phi(\beta, h)$ equals the "annealed" one, if for instance β is small. The annealed free energy is just

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_h Z_{n,\beta,\sigma} \tag{3.3}$$

which by Jensen dominates $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_h \log Z_{n,\beta,\sigma} = \Phi(\beta, h)$. It is however easily checked that (3.3) is $> \beta h$ for *all* $\beta > 0, 0 \leq h < 1$. In fact

$$\mathbb{E}_h Z_{n,\beta,\sigma} = \cosh(\beta)^n E \exp\left[\sum_{i=1}^n \log\left(1 + h\Delta_i \tanh(\beta)\right)\right],$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_h Z_{n,\beta,\sigma} = \log(\cosh(\beta) + h\sinh(\beta)) > h\beta$$

for $h < 1$. The last inequality is easily checked: The two sides agree at $h = 1$, and then the l.h.s. is concave and bigger than the r.h.s. at $h = 0$.

Therefore, we are never able to prove $\Phi(\beta, h) = \beta h$ in this way except for $h = 1$, where it is trivial. However, a slight modification of the above argument leads to

Proposition 3.75. a) $\Phi(\beta,h)=\beta h$ *if*

$$h \geq \frac{\cosh(2\beta)-1}{\sinh(2\beta)}.$$

b) *If* (β,h) *satisfies* $\Phi(\beta,h)=\beta h$*, then* $\Phi(\beta',h')=\beta' h'$ *if* $\beta'>\beta, h'>h$*, and*

$$e^{2(\beta'-\beta)}\left[\frac{h'-h}{1-h}e^{-2\beta}+\frac{1-h'}{1-h}\right]\leq 1 \tag{3.4}$$

Proof. a) The trick is to take a (trivial) part out of Z which is handled in a "quenched way", and discuss the other part by an estimate of the type discussed above.

$$\begin{aligned} Z_{n,\beta,\sigma} &= E\exp\left[\beta\sum_{i=1}^{n}\sigma_i\Delta_i\right] \\ &= \exp\left[\beta\sum_{i=1}^{n}\sigma_i\right]E\exp\left[\beta\sum_{i=1}^{n}\sigma_i(\Delta_i-1)\right]. \end{aligned}$$

Evidently, $\frac{1}{n}\log$ of the first factor goes to βh, so we have to estimate the second, which we do by Jensen.

$$\begin{aligned} &\mathbb{E}_h\log E\exp\left[\beta\sum_{i=1}^{n}\sigma_i(\Delta_i-1)\right] \\ &\qquad\leq \log E\prod_{i=1}^{n}\left[\cosh(\beta(\Delta_i-1))+h\sinh(\beta(\Delta_i-1))\right] \\ &\qquad\leq \log\left[(\cosh(2\beta)-h\sinh(2\beta))\vee 1\right]\leq 0, \end{aligned}$$

if $h\geq(\cosh(2\beta)-1)/\sinh(2\beta)$.

b) This comes by a modification of the above argument. If $h'>h$, we denote by (σ_i) the signs distributed according to $\mathbb{P}_h$, and then we choose, conditionally on (σ_i) independent τ_i with $\tau_i=2$ w.p. $(h'-h)/(1-h)$ if $\sigma_i=-1$, and $\tau_i=0$ otherwise. Then $\sigma_i+\tau_i$ is distributed according to $\mathbb{P}_{h'}$. We just write $\mathbb{P}$ for the joint law. Then if $\beta'>\beta$

$$\begin{aligned} &\mathbb{E}\log E\exp\left[\beta'\sum_{i=1}^{n}(\sigma_i+\tau_i)(\Delta_i-1)\right] \\ &\quad=\mathbb{E}\left\{\log E\exp\left[\beta\sum_{i=1}^{n}\sigma_i(\Delta_i-1)\right]\right. \\ &\qquad\left.\times\,\mathbb{E}\left(\exp\left[\sum_{i=1}^{n}(\beta'-\beta)\sigma_i+\beta'\tau_i)(\Delta_i-1)\right]\middle|\,\sigma\right)\right\}. \end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left(\exp\left[\sum_{i=1}^{n} (\beta' - \beta)\sigma_i + \beta'\tau_i)(\Delta_i - 1) \right] \middle| \sigma \right) \\
&= \prod_{i=1}^{n} \exp\left[(\beta' - \beta)\sigma_i(\Delta_i - 1)\right] \\
&\quad \times \left\{ 1_{\sigma_i=1} + 1_{\sigma_i=-1} \left[\frac{h' - h}{1-h} \mathrm{e}^{2\beta(\Delta_i - 1)} + \frac{1-h'}{1-h} \right] \right\} \\
&\leq 1
\end{aligned}$$

for any choice of $(\sigma_i), (\Delta_i)$ as soon as (3.4) is satisfied. From this, b) follows.

Theorem 3.76. *For any $\beta > 0$ there exists $h_c(\beta) \in (0,1)$ such that $\Phi(\beta, h) > 0$ for $h < h_c(\beta)$, and $\Phi(\beta, h) = 0$ for $h \geq h_c(\beta)$. The function $\beta \to h_c(\beta)$ has the following properties:*

a) *$\beta \to h_c(\beta)$ is continuous and non-decreasing.*
b) *$\lim_{\beta \to \infty} h_c(\beta) = 1$.*
c) *$\limsup_{\beta \downarrow 0} h_c(\beta)/\beta \leq 1$.*
d) *$\liminf_{\beta \downarrow 0} h_c(\beta)/\beta > 0$.*

Before I start with the proof, some comments. The largest part of [15] had actually been spent in proving that $\lim_{\beta \downarrow 0} h_c(\beta)/\beta$ exists in $(0, \infty)$, a fact which had been predicted in the physics literature, and "identifying" in a sense this limit. The proof of this is however rather delicate for reasons I will indicate below. I cannot give the details of this here. So I stick with proving just c), d).

One might think that a discussion of the $\beta \to 0$ limit leads to a perturbation expansion, but this is not the case. The tangent of $h_c(\beta)$ at $\beta = 0$ seems to be a complicated object and we have not much information about it except that it exists.

Proof of Theorem 3.76. From Proposition 3.75, we see that for any $\beta \geq 0$, if h is sufficiently close to $1, \Phi(\beta, h) = \beta h$. Remark also that

$$\Phi(\beta, h) - \beta h = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_h \log\ E \exp\left[\beta \sum_{i=1}^{n} \sigma_i (\Delta_i - 1) \right],$$

and from this we see that $\Phi(\beta, h) - \beta h$ is non-increasing in h. As it is also continuous, we conclude that there exists $h_c(\beta) \in [0, 1)$ such that $\Phi(\beta, h) > \beta h$ if $h < h_c(\beta)$ and $\Phi(\beta, h) = \beta h$ if $h \geq h_c(\beta)$.

That $\beta \to h_c(\beta)$ is nondecreasing follows from the convexity of Φ in β which implies (together with $\Phi(0, h) = 0$) that if $\Phi(\beta, h) - \beta h > 0$, then $\Phi(\beta', h) - \beta' h > 0$ for all $\beta' > \beta$.

The continuity of h_c follows from Proposition 3.75 which implies that there are no upward jumps. Therefore, we have proved a).

c) is also a consequence of Proposition 3.75 a) and we are left with proving b) and d). Remark that d) in particular implies $h_c(\beta) > 0$ for all $\beta > 0$.

The proofs of b) and d) of course just require a lower bound for Φ. We follow a standard approach for proving lower bounds in large deviation theory, by changing the measure. Although our new measure is good enough to produce the lower bounds we are looking for, the reader will realize that (in contrast to the situation we had in Chapter 2) our reference measures are not the "correct" ones.

The first observation is that the only relevant information needed about the random walk are the successive return times to 0: $\eta_0 = 0, \eta_{j+1} = \inf\{i > \eta_j : \omega_i = 0\}$. Set $\Delta\eta_j = \eta_j - \eta_{j-1}$. The $\Delta\eta_j$ are i.i.d. random variables. As before, we put $q(\ell) = P(\Delta\eta_j = l)$. The η_j define random points, or more formally random variables Z_k, $k \in \mathbb{N}$, where $Z_k = 1$ if $k = \eta_j$ for some j, and $Z_k = 0$, otherwise.

We define $\mathcal{F}_n = \sigma(Z_k : k \leq n)$. Our partition function has a trivial recasting in terms of the (Z_k) process by just integrating out the sign of the excursion between two successive returns to 0. Put $\psi(x) = \log\cosh(x)$, and

$$H_{n,\beta}(Z,\sigma) = \sum_{j=1}^{\tau_n} \psi\left(\beta \sum_{i=\eta_{j-1}+1}^{\eta_j - 1} \sigma_i\right) + \psi\left(\beta \sum_{i=\eta_{\tau_n}+1}^{n} \sigma_i\right),$$

where $\tau_n = \max\{k \leq n : \eta_k \leq n\}$. Then evidently

$$Z_{n,\beta,\sigma} = E_n^Z \exp\left[H_{n,\beta}(Z,\sigma)\right],$$

where E_n^Z refers to taking the expectation for the sequence $Z_0 = 1$, $Z_1, \ldots, Z_n$. Of course, we can introduce now also the Gibbs measure on the Z-sequence (for fixed σ) with this Hamiltonian. This however is a very complicated object, with a *very* complicated many body interaction (for fixed σ). Despite of this fact, we perform just a very simple change of measure, changing just the distribution f to

$$q^{\gamma}(\ell) = \frac{1}{1-\gamma} q(\ell)(\sqrt{1-\gamma^2})^{\ell},$$

$\gamma > 0$. We write $P_n^{Z,\gamma}$ for the corresponding distribution of $Z_0, \ldots, Z_n$. Then by Jensen

$$\begin{aligned} Z_{n,\beta,\sigma} &= E_n^{Z,\gamma} \exp\left(H_{n,\beta}(Z,\sigma) - \log\frac{dP_n^{Z,\gamma}}{dP_n^Z}\right) \\ &\geq \exp(E_n^{Z,\gamma} H_{n,\beta}(Z,\sigma) - k(P_n^{Z,\gamma}|P_n^Z)), \end{aligned} \tag{3.5}$$

where $k(\mu|\nu)$ is the usual Kullback-Leibler relative entropy. We now only have to choose the appropriate γ for getting the desired lower bounds. I stick to proving d) in Theorem 3.76, i.e. prove the bound for $\beta \sim 0$. The $\beta \to \infty$ is not difficult either, and I will leave this to the reader.

Although this very crude argument is giving a lower bound proving that h_c has a positive slope at 0, it seems clear that it could not produce the correct tangent (which we know to exist by considerations I will indicate below).

First, we estimate the entropy

$$\frac{dP_n^{Z,\gamma}}{dP_n^Z} = \prod_{j=1}^{\tau_n} \frac{q^\gamma}{q}(\eta_j - \eta_{j-1}) \frac{\sum_{\ell > n-\eta_{\tau_n}} q^\gamma(\ell)}{\sum_{\ell > n-\eta_{\tau_n}} q(\ell)}$$
$$\leq (1-\gamma)^{-\tau_n - 1}(1-\gamma^2)^{n/2}.$$

Therefore

$$k(P_n^{Z,\gamma}|P_n^Z) \leq -\log(1-\gamma)E_n^\gamma(\tau_n + 1) + \frac{n}{2}\log(1-\gamma^2).$$

By the optional sampling theorem, we get

$$\lim_{n\to\infty} \frac{1}{n} E_n^\gamma(\tau_n+1) = \frac{\gamma}{1+\gamma},$$

and therefore

$$\limsup_{n\to\infty} \frac{1}{n} k(P_n^{Z,\gamma}|P_n^Z) \leq -\frac{\gamma}{1+\gamma}\log(1-\gamma) + \frac{1}{2}\log(1-\gamma^2). \tag{3.6}$$

On the other hand, we have

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}E_n^{Z,\gamma} H_n(Z,\sigma) \geq \liminf_{n\to\infty} \frac{1}{n} E_n^{Z,\gamma}\left(\sum_{j=1}^{\tau_n} \mathbb{E}\psi\left(\beta \sum_{i=\eta_{j-1}+1}^{\eta_j} \sigma_i\right)\right)$$
$$= \frac{\gamma}{\gamma+1} E_n^{Z,\gamma}\mathbb{E}\psi\left(\beta\sum_{i=1}^{\eta_1}\sigma_i\right)$$

Now, we would like to apply Jensen to the convex function ψ, but estimating just $\mathbb{E}\psi(\beta(\sum_{j=1}^{\eta_j}\sigma_j) \geq \psi(\beta\mathbb{E}(\sum_{j=1}^{\eta_j}\sigma_j)) = \psi(\beta\eta_1 h)$ would kill the baby. In fact, this would mean that we just use that the σ_j have a average density of $+1$ of the right rate, and would not use the fluctuations. The fact that there is a positive slope of the h_c-line is coming from the fluctuations of the σ_i. We use a slightly more clever argument, by applying Jensen to the conditioned law $\left(\mathbb{P}\cdot \middle| \sum_{j=1}^{\eta_1}\sigma_j \geq \eta_j h\right)$. Remark that $\mathbb{P}(\sum_{j=1}^{\eta_1}\sigma_j \geq \eta_1 h) \geq c_1(h) > 0$, where $c(h) \to 1/2$ for $h \to 0$. Therefore

$$\mathbb{E}\psi(\beta\sum_{j=1}^{\eta_1}\sigma_j) \geq c_1(h)\psi\left(\beta\eta_1 h + \beta\mathbb{E}\left(\sum_{j=1}^{\eta_1}(\sigma_j - h)\middle|\sum_{j=1}^{\eta_1}(\sigma_j-h) \geq 0\right)\right)$$
$$\geq c_1(h)\psi(\beta\eta_1 h + \beta c_2(h)\sqrt{\eta_1}),$$

where again, $c_2(h)$ stays bounded away from 0 as $h \to 0$. Combining this with (3.6) and (3.5), we get

$$\begin{aligned}\Phi(\beta,h) &\geq \frac{\gamma c_1(h)}{1+\gamma} E_n^{Z,\gamma} \psi(\beta\eta_1 h + \beta c_2(h)\sqrt{\eta_1}) \\ &\quad + \frac{\gamma}{1+\gamma}\log(1-\gamma) - \frac{1}{2}\log(1-\gamma^2) \\ &= \frac{\gamma c_1(h)}{1-\gamma^2}\sum_{\ell} q(\ell)(1-\gamma^2)^{\ell/2}\psi(\beta h\ell + \beta c_2(h)\sqrt{\ell}) \\ &\quad + \frac{\gamma}{1+\gamma}\log(1-\gamma) - \frac{1}{2}\log(1-\gamma^2).\end{aligned}$$

We now let $\beta \to 0, \gamma = b\beta, h = a\beta$; a, b to be chosen later on. Then we get

$$\begin{aligned}&\liminf_{\beta\to 0} \frac{1}{a\beta^2}\Phi(\beta,\beta a) \\ &\geq \frac{c_1 b}{a}\int_0^\infty \frac{dx}{x^{3/2}} \exp\left[-\frac{b^2}{2}x\right]\psi(ax + c_2\sqrt{x}) - \frac{b^2}{2a},\end{aligned}$$

where we have used that $q(\ell) \sim c_1\ell^{-3/2}$, if $\ell \to \infty$ along even numbers. We can still choose b at our liking, so we take $b = Ka$, and let $a \to 0$. Then

$$\begin{aligned}&\liminf_{a\to 0}\int_0^\infty \frac{dx}{x^{3/2}}\exp\left[-\frac{Ka^2}{2}x\right]\psi(ax + c_2\sqrt{x}) \\ &\geq \int_0^1 \frac{dx}{x^{3/2}}\psi(c_2\sqrt{x})dx > 0.\end{aligned}$$

Therefore, as K is arbitrary

$$\liminf_{a\to 0}\liminf_{\beta\to 0}\frac{1}{a\beta^2}\Phi(\beta,\beta a) = \infty.$$

In particular,

$$\liminf_{\beta\to 0}\frac{1}{a\beta^2}\Phi(\beta,\beta a) > 1$$

if a is small enough. This proves part d) of the Theorem.

As remarked, I leave part b) to the reader.

The $\beta \sim 0$ case is actually somewhat puzzling at first sight. Let us look at the symmetric $h = 0$ case. The above argument for the lower bound shows also that $\Phi(\beta, 0)$ is of order β^2. It is in fact true that $\lim_{\beta\to 0}\beta^{-2}\Phi(\beta,0)$ exists and is positive (see Proposition 3.79 below). Choosing γ in the above proof of order β gives a lower bound of the correct order β^2 (but not the correct constant). The procedure for this lower bound is somewhat naive. Our reference measure just shortens the excursions from 0 of the random walk by changing the distribution q. This happens completely regardless what the σ's are. The σ's play only a rôle for choosing the signs of the excursion. This aspect is somewhat hidden in the proof above, as we have integrated out these signs immediately, but the effect can easily be reconstructed. If we have an excursion, of length ℓ, say, and S_ℓ is the sum over the σ's, then the excursion chooses the "oily side" with probability $e^{\beta S_\ell}/(e^{\beta S_\ell} + e^{-\beta S_\ell})$. Now, as remarked, it is certainly *not* the case that this measure on paths describes the true Gibbs measure accurately, but as the lower bound obtained using this measure is of the correct order, at least for $\beta \sim 0$, we might guess that the true Gibbs measure has qualitatively about similar properties. For the correct Gibbs measure, the σ's obviously influence the place where the excursions occur, and not just the sign as in the above strategy for the lower bound.

For $\beta \to 0$ the problem rescales to a problem on the Brownian motion. Let $\{\omega_t\}_{t\geq 0}$ be a standard Brownian, playing the rôle of the random walk, and $\{\sigma_t\}$ an independent Brownian motion whose derivative plays the rôle of the random environment. The model we would then naturally have is a transformation of the law of the first motion by $\{\sigma_t\}$, a coupling parameter $\beta > 0$, and a drift parameter h:

$$\hat{P}_{T,\beta,h,\sigma}(d\omega) = \exp\left[\beta \int_0^T \operatorname{sign}(\omega_s)(d\sigma(s) + hds)\right] P_T(d\omega)/Z_{T,\beta,h,\sigma},$$

where $P_T(d\omega)$ is standard Wiener measure, and we define

$$\widehat{\Phi}(\beta, h) = \lim_{T\to\infty} \frac{1}{T} \log Z_{T,\beta,h,\sigma}.$$

It is in fact not difficult to prove that $\widehat{\Phi}(\beta, h)$ exists in $(0,\infty)$ and has similar properties as Φ introduced in Proposition 3.73. In contrast to Φ, $\widehat{\Phi}$ has a very simple rescaling property which just comes from Brownian rescaling: Setting $\sigma(t) = \varrho\tilde{\sigma}(t/\varrho^2)$, we get

$$\begin{aligned} Z_{T,\beta,h,\sigma} &= E\exp\left[\beta \int_0^T \operatorname{sign}(\omega_t)(d\sigma(t) + hds)\right] \\ &= E\exp\left[\varrho\beta \int_0^{T/\varrho^2} \operatorname{sign}(\omega_{s\varrho^2})(d\tilde{\sigma}(s) + \varrho hds)\right]. \end{aligned}$$

Remark now that if we put $\tilde{\omega}_s = \omega_{s\varrho^2}/\varrho$, the sign is not influenced by the scaling, and therefore, we get

$$\widehat{\Phi}(\beta, h) = \frac{1}{\varrho^2}\widehat{\Phi}(\varrho\beta, \varrho h). \tag{3.7}$$

If we put $\Lambda(h) = \widehat{\Phi}(1, h)$, we therefore have

$$\widehat{\Phi}(\beta, h) = \beta^2 \Lambda(h/\beta).$$

It is not difficult to prove along the same lines as in the proof of Theorem 3.76 the following

Proposition 3.77. *There is a critical value* $\tilde{h}_c > 0$ *such that*

a) $\Lambda(h) = h$ *if* $h \geq \tilde{h}_c$.
b) $\Lambda(h) > h$ *if* $h < \tilde{h}_c$.

There seems to be no decent representation of h_c for instance as some solution of a variational problem. Crude arguments like the ones before just show

- Λ is continuous
- $\Lambda(h) \geq h$ for all h
- $\Lambda(h) = h$ for h large enough
- $\Lambda(0) > 0$,

which imply the above proposition. It is however unclear if arguments of the type described before are able to characterize the correct value $\widetilde{h}_c$.

Given the value of $\widetilde{h}_c$, the rescaling property shows that the phase separation line for the continuous model in the (β, h) plane is just a straight line.

The following result is quite plausible but the proof is very delicate and cannot be given here.

Theorem 3.78. *The phase separation line* $h_c(\beta)$ *(for the discrete model) is differentiable at* 0 *with derivative* $\tilde{h}_c$.

I close this section with some comments about the above result.

We expect (but did not prove) that this tangent $\tilde{h}_c$ is quite universal and does not much depend on special properties of the discrete model. Evidently, changing for instance the law of the random walk or the environment would lead to different phase separation curves, but the tangent at 0 should not depend on specific properties of the model.

We believe that there cannot be a particularly easy derivation of the theorem from standard invariance principles.

There is an intermediate result which is easier to prove, namely convergence of the free energy.

Proposition 3.79.

$$\lim_{\beta \to 0} \frac{1}{\beta^2} \Phi(\beta, h\beta) = \Lambda(h).$$

The proof of this proposition considerably simpler than that of Theorem 3.78, but it is not easy either. The delicacy is that we do not really have good control of the true Gibbs measure. Essentially, one has to show that this Gibbs measures goes over, in the $\beta \to 0$ limit to the corresponding Gibbs measure for the continuous model, which is as complicated (or even more so). Of course, we are not able to prove that on the level of Gibbs measures, but only on the level of free energies.

Unfortunately, the proposition does not even quite imply the Theorem 3.78, but only half of it. If $h < \tilde{h}_c$, then $\Lambda(h) > h$, and therefore $\Phi(\beta, \beta h) > \beta^2 h$ for small enough β. This shows that the slope of $h_c(\beta)$ at $\beta = 0$ is at least $\tilde{h}_c$. However, if $h > \tilde{h}_c$ then the Proposition implies only that

$$\lim_{\beta \to 0} \frac{1}{\beta^2} \Phi(\beta, h\beta) = h.$$

This of course does not imply that $\Phi(\beta, h\beta)/\beta^2$ equals h for small enough $\beta > 0$, which is required for the proof of Theorem 3.78. The proof of the last fact uses repeated applications of much more complicated versions "semiannealed" trick, as in the proof of Proposition 3.75, but in only very implicitly defined situations. I think that the proof is of some methodological interest, but it is far too involved to be presented here.

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